Learning in Online Advertising

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Abstract

Prior literature on pay-per-click advertising assumes that publishers know advertisers' click-through rates (CTR). This information, however, is not available when a new advertiser first joins a publisher. The new advertiser's CTR can be learned only if its ad is shown to enough consumers, i.e., the advertiser wins enough auctions. Since publishers use CTRs to calculate payments and allocations, the lack of information about a new advertiser can affect the advertisers' bids. Using a game theory model, we analyze advertisers' strategies, their payoffs, and the publisher's revenue in a learning environment. Our results indicate that a new advertiser always bids higher (sometimes above valuation) in the beginning. The incumbent advertiser's strategy depends on its valuation and CTR. A strong incumbent increases its bid to deter the publisher from learning the new advertiser's CTR, whereas a weak incumbent decreases its bid to facilitate learning. Interestingly, the publisher may benefit from not knowing the new advertiser's CTR because its ignorance could induce advertisers to bid more aggressively. Nonetheless, the publisher's revenue sometimes decreases because of this lack of information. The publisher can mitigate this loss by lowering the reserve price of, offering advertising credit to, or boosting the bids of new advertisers.

1 Introduction

Online advertising, with an annual spending of over $100B, has become the largest category of advertising in the US.[1] Online advertising inventory is sold using two pricing models: performance-based and impression-based. In performance-based (e.g., pay-per-click) pricing, an advertiser pays only if a consumer completes a pre-defined action (e.g., a click). In impression-based pricing, the advertiser pays for its ad being shown to a consumer, regardless of whether the impression leads to an action.

Understanding how an ad performs (e.g., how likely a consumer will take an action after viewing an ad) is crucial for publishers in performance-based pricing, and for advertisers in impression-based pricing. For example, in pay-per-click pricing, it is more profitable for a publisher to accept a payment of $1 per click for an ad with click-through rate (CTR) 10%, for an expected revenue $0.10 per impression, than a payment of $2 per click for an ad with CTR 4%, for an expected revenue $0.08 per impression. Similarly, the probability of action affects an advertiser's willingness to pay (WTP) for an impression in impression-based pricing. The advertiser is willing to pay more per impression if it knows that the impression leads to a desired outcome with a higher probability.

Previous literature on online advertising primarily assumes that the probability of the pre-defined action (e.g., CTR) is known to advertisers and publishers (Edelman et al., 2007; Katona and Sarvary, 2010;

[1] `https://content-na1.emarketer.com/us-ad-spending`

Jerath et al., 2011). In practice, however, advertisers and publishers have to *learn* this probability. For example, when a new advertiser joins the market, or when an existing advertiser revamps its ad campaign, the CTRs of its ads are typically unknown to the publisher, other advertisers, and the advertiser itself. They can at best have an expectation of the CTR based on a few observable characteristics of the advertiser.[2] The actual CTR becomes known only when the ads are displayed to consumers enough number of times such that sufficient impression and click data become available. In other words, learning is asymmetric: *participating* in advertising auctions is not sufficient for the advertiser's CTR to be learned; the advertiser has to *win* advertising auctions sufficiently many times before the publisher and the advertisers can learn its CTR.

The learning dynamic can affect advertisers' and publishers' strategies in the market. In particular, winning in an advertising auction has two effects on an advertiser's payoff. First, the advertiser receives an immediate value from showing its ad to a consumer (the direct effect). Second, winning reveals information about the performance of the ad to both the advertiser and the publisher (the indirect effect); this improves the advertiser's and the publisher's estimate of the true CTR of the ad. In performance-based pricing, this ad-performance information is used by the publisher to determine the pricing and allocation of an ad slot, and in impression-based pricing, it is used by the advertiser to determine the advertiser's WTP. While the previous literature has primarily studied the direct effect of winning in an advertising auction, our paper focuses on the indirect effect.

These two effects give rise to interesting trade-offs for advertisers when a new advertiser joins the publisher. We illustrate these trade-offs in the following example.

Example. Suppose an advertiser, A, is the only advertiser bidding on an advertising slot of publisher P. Suppose that the slot is sold in a pay-per-click second price auction, A's bid is \$1 per click, and its CTR is 15%. Assume that B is a new advertiser who wants to advertise on the same slot. B's bid is also \$1 per click, but its CTR is not known to anybody at the time of entry. For the initial auctions, P assigns an average CTR estimate (e.g., based on the performance of advertisers with similar characteristics) of, say, 10%.

However, P can eventually learn the new advertiser's CTR after sufficient impression and click data

[2]For instance, in pay-per-click pricing, Google assigns an average Quality Score to new advertisers based on the performances of other advertisers using the same keyword. See `https://searchengineland.com/didnt-know-recent-quality-score-changes-259559`.

for the new advertiser become available. Furthermore, B can facilitate this learning process by bidding aggressively and thereby winning in the early rounds. Doing so allows P to observe more click data for B's ad which would in turn allow P to more accurately estimate B's true CTR.

Importantly, in pay-per-click pricing, P's estimate of B's CTR directly affects the payment and allocation of the advertisers. This is because publishers use *effective bids*, computed as advertisers' submitted bids multiplied by their expected CTRs, to calculate payment and allocation.[3] Given this, would B prefer to have its CTR learned by P quickly or not?

If B privately knew its true CTR, then the answer would be evident. For example, if it knew that its true CTR is 20% (i.e., higher than P's estimate), then B would unambiguously prefer P to quickly update its CTR from the 10% estimate to the true 20%. The reason is that updating its CTR to a higher value would not only make B's future effective bid more competitive against the existing advertiser A, but also lower B's cost-per-click when it wins. In particular, with its \$1 bid and 20% CTR, B will outrank A's effective bid of \$1 $\times$ 15% and win the auction for a cost-per-click of \$0.75; it would have lost the auction to A had its CTR remained at the average of 10% (see Table 1). Conversely, B's incentive to facilitate P's learning its CTR would diminish if B knew its true CTR is *lower* than P's prior estimate. In this case, B's long-term payoff would decrease if its low CTR is learned quickly. In sum, B prefers P to update B's CTR estimate more quickly (slowly) if it privately knows that its CTR is higher (lower) than P's prior estimate.

	Advertiser B's CTR		
	Not known (CTR=10%)	Known (CTR=20%)	Known (CTR=5%)
B	Has to bid (and pay) $\$1 \times 15\%/10\% = \$1.5$ to win	Wins at cost-per-click $\$1 \times 15\%/20\% = \$0.75$	Has to bid (and pay) $\$1 \times 15\%/5\% = \$3$ to win
A	Wins at cost-per-click $\$1 \times 10\%/15\% = \$0.66$	Has to bid (and pay) $\$1 \times 20\%/15\% = \$1.33$ to win	Wins at cost-per-click $\$1 \times 5\%/15\% = \$0.33$

Table 1: When the Publisher Knows vs. Does Not Know New Advertiser's CTR

In reality, however, when B first enters the market, it does *not* know whether its true CTR is lower or higher than an average advertiser with similar characteristics. Therefore, it is not clear whether the new advertiser B should increase or decrease its bid to accelerate or slow down P's learning process if B wants to maximize its profit.

[3] In practice, effective bids can also include other factors such as landing page experience and advertiser's reputation; however, for the purpose of this example, we only consider the expected CTR and the submitted bid that are the two most important elements of effective bids.

Similarly, for the existing advertiser A, P's learning the new advertiser B's CTR can be a double-edged sword. If B's CTR turns out to be higher than the estimated average, then A may lose the ad slot; if it turns out to be lower, A can win the auction at a lower cost-per-click than when B's CTR is not known to P (from \$0.66 to \$0.33 in Table 1). Again, given that the existing advertiser A can facilitate (hinder) P's learning process by decreasing (increasing) its bids when B joins, it is not clear which bidding strategy would maximize its profit.

In this paper, we study how the learning incentives affect the advertisers' and the publisher's strategies. We use a game-theoretic model to analyze advertisers' and publisher's strategies in a learning environment. To facilitate exposition, in the main body of the paper, we assume the publisher uses performance-based pricing, which currently accounts for 62% of the online advertising market in the US,[4] and use pay-per-click terminology. In the extensions, we show that our results apply to pay-per-impression pricing model as well. We are interested in answering the following research questions.

1. Does a new advertiser (entrant) benefit from its CTR being learned by the publisher? How does this affect the entrant's bidding strategy?

2. Does an existing advertiser (incumbent) benefit from the publisher learning the CTR of the entrant? How does this affect the incumbent's bidding strategy?

3. How does the lack of information about a new advertiser's CTR affect the publisher's revenue? How do learning incentives affect the publisher's optimal strategy?

In answering the first set of questions, we show that a new advertiser's expected payoff when its CTR is learned by the publisher is higher than when it is not. The higher payoff incentivizes the new advertiser to bid aggressively to accelerate the learning process. As a result, the entrant should always bid higher (sometimes even above its valuation) in the beginning when its CTR is unknown to the publisher, than in the long run after its CTR becomes known. This finding is in line with what industry experts commonly recommend new advertisers regarding starting bids — namely, bid aggressively "into high positions" and "make adjustments after [accumulating] data." Despite the risk of paying a high initial cost, the experts explain that bidding high and thereby attaining top positions early on could help improve the advertisers' long-run profits.[5]

[4] `http://totalaccess.emarketer.com/chart.aspx?r=219092`
[5] `https://searchengineland.com/4-ways-to-determine-your-your-starting-bids-144616`.

Our result indicate that even for advertisers whose long-run equilibrium cost-per-click is low, the initial cost-per-click (at the time of joining the market) may be above their valuation. In other words, advertisers should be prepared to lose money in the beginning when they start advertising with a publisher for the first time. Moreover, they should not be discouraged from using that publisher even if the initial cost-per-clicks are higher than their WTP.

In answering the second set of questions, we find that an incumbent's response to an entrant joining the auction depends on the incumbent's CTR. If the incumbent's CTR is high, the incumbent bids aggressively to impede the entrant's CTR from being learned by the publisher. This is because an incumbent with a high CTR does not want to risk earning a low margin (or worse, losing its ad slot) in the event the entrant's CTR turns out to be high.

This "preemptive" strategy, however, is too expensive for an incumbent with a low CTR. As we show, an incumbent with a low CTR lowers its bid when an entrant joins, so that the entrant's CTR is learned more quickly. Intuitively, competing with an advertiser whose CTR is unknown is too costly for the weak incumbent; by accelerating the learning process, the incumbent hopes that the entrant's CTR will turn out to be lower than expectation.

In answering the third set of questions, interestingly, we find that the publisher may benefit from *not* knowing the new advertiser's CTR. The intuition is that the entrant, and sometimes the incumbent as well, bids more aggressively when the entrant's CTR is not known, which increases the publisher's revenue. Under certain conditions, however, the publisher's ignorance could also hurt its revenue. For instance, if the entrant's CTR is high, the publisher misses clicks (and hence opportunities for earning higher revenue) by not displaying the entrant's ad in the beginning. The negative effect becomes more pronounced when the incumbent's CTR is high because a strong incumbent bids aggressively to mask the entrant's CTR. This deters the publisher from learning the entrant's potentially high CTR.

We find that the publisher can mitigate the loss of not knowing the entrant's CTR by reducing the reserve price of the entrant. By reducing the reserve price, the publisher increases the probability of the entrant winning in the auction, thereby increasing the probability of learning the entrant's CTR. Furthermore, we characterize the optimal mechanism and show that, first, in the presence of learning considerations, a variation of the standard second-price auction with optimal reserve prices is sufficient to achieve the optimal revenue. Second, it is optimal for the publisher to favor the entrant in the beginning, before the entrant's CTR is learned. This manifests in a lower optimal reserve price of the entrant when the publisher does not know the entrant's CTR than when it knows.

In addition, we discuss alternative mechanisms that can help the publisher mitigate its loss of not knowing the entrant's CTR. For example, Google provides $75 ad credit to new advertisers when they spend $25 on AdWords.[6] Facebook also offers ad credit to new accounts that have a sufficiently high audience engagement on their pages.[7] While these programs have traditionally been viewed as promotions to attract new advertisers, our research reveals new strategic incentives beyond new customer acquisition that motivate publishers to offer ad credit.

Theoretical Contribution. While, from a managerial point of view, our work sheds light on advertisers' and publishers' strategies regarding new entries, we also want to highlight two unique aspects of our model from a theoretical point of view. First, in the context of online advertising, we study the *transition* of a game from an incomplete information game to a full information one. While the previous literature on online advertising assumes that the game is either always full information (e.g., Edelman et al., 2007) or always incomplete information (e.g., Edelman and Schwarz, 2010), in practice, the level of information is constantly changing. Our paper takes a first step towards bridging this gap by analyzing the transition.[8] We show that the advertisers' and the publishers' strategies regarding the transition are qualitatively distinct from those in full information and incomplete information games.

Second, our analysis demonstrates how some of the standard results from learning theory may be reversed when the subjects of learning are not as "passive" as commonly assumed in the literature (e.g., Gittins and Jones, 1979; Katehakis and Veinott, 1987). For instance, exploration-exploitation trade-off from standard learning theory suggests that knowing less about new advertisers would only hurt the publisher's revenue because the publisher must then learn about new advertisers through costly exploration. In contrast, our model shows that the publisher may be better off knowing less about the new advertiser due to the advertisers' strategic responses during the publisher's learning process. In other words, when the subjects are strategic agents, exploration could be *profitable* for the learner.

The rest of this paper is structured as follows. First, we discuss related literature. In Section 2, we

[6] `https://www.google.com/ads/adwords-coupon.html`

[7] `http://www.digitalsitemap.com/free-facebook-ad-coupon/`

[8] In fact, since new advertisers constantly join this market, and even existing advertisers frequently revamp their campaigns, change their ad copies and landing pages, or change their ad agencies altogether, one could argue that this market is always in transition.

present the model. We analyze the model and discuss advertisers' strategies in Section 3. The publisher's optimal strategy is discussed in Section 4. We explore extensions of the main model in Section 5 to establish the robustness of our main results, and conclude in Section 6. All proofs are relegated to Appendix A.

Related Literature

Our work contributes to the vast literature on display advertising. Empirical works in this area have assessed the effectiveness of display advertising in various contexts. Lambrecht and Tucker (2013) demonstrate that retargeting may not be effective when consumers have not adequately refined their product preferences. Hoban and Bucklin (2015) find that display advertising increases website visitations for a large segment of consumers along the purchase funnel, but not for those who had visited before. Bruce et al. (2017) examine the dynamic effects of display advertising and show that animated (vs. static) ads with price information are the most effective in terms of consumer engagement. On the theoretical front, Sayedi et al. (2018) study advertisers' bidding strategies when publishers allow advertisers to bid for exclusive placement on the website. Sayedi (2018) analyzes the interaction between selling ad slots through real-time bidding and selling through reservation contracts. Zhu and Wilbur (2011) and Hu et al. (2016) study the trade-offs involved in choosing between "cost-per-click" and "cost-per-action" contracts. Berman (2016) explores the effects of advertisers' attribution models on their bidding behavior and their profits. Kuksov et al. (2017) study firms' incentives in hosting the display ads of their competitors on their websites.

Within online advertising, the increasing prevalence of search advertising has motivated a growing body of empirical (e.g., Rutz and Bucklin, 2011; Yao and Mela, 2011; Haruvy and Jap, 2018) and theoretical papers. Katona and Sarvary (2010) and Jerath et al. (2011) study advertisers' incentives in obtaining lower vs. higher positions in search advertising auctions. Sayedi et al. (2014) investigate advertisers' poaching behavior on trademarked keywords, and their budget allocation across traditional media and search advertising. Desai et al. (2014) analyze the competition between brand owners and their competitors on brand keywords. Lu et al. (2015) and Shin (2015) study budget constraints, and budget allocation across keywords. Zia and Rao (2017) look at the budget allocation problem across search engines. Wilbur and Zhu (2009) find the conditions under which it is in a search engine's interest to allow some click fraud. Cao and Ke (2017) model a manufacturer and retailers' cooperation in search advertising and show how it affects intra- and inter-brand competition. Amaldoss et al. (2015a) show how a search engine can increase its profits and also improve advertisers' welfare by providing first-page bid estimates. Berman and Katona (2013) study the impact of search engine optimization, and Amaldoss et al. (2015b) analyze the effect of keyword management costs on advertisers' strategies. Katona and Zhu (2017) show how quality scores can incentivize advertisers to invest in their landing pages and to improve their conversion rates.

Following Edelman et al. (2007), by arguing that players learn each others' types after playing the game repeatedly, the vast majority of this literature uses a full information setup to model search advertising auctions. There are a few papers (e.g., Amaldoss et al., 2015a,b; Edelman and Schwarz, 2010) that use an incomplete information setting for modeling search advertising. In these papers, however, the game remains an incomplete information game; i.e., players do not learn each others' types. To the best of our knowledge, our paper is the first on online advertising to model the learning process, wherein the game starts as an incomplete information game and, if a new advertiser's type is learned, transitions to a full information game.

Parts of our model may resemble the literature on games with asymmetric information. For instance, in Jiang et al. (2011), a seller may want to hide its type from a publisher by pooling with another type. Despite some similarities, our paper differs in that we do not model information asymmetry. Although we allow players take certain actions to facilitate or hinder the revelation of information, those actions do not signal their types. Furthermore, in signaling games, players mimic other players' strategies in order to hide or reveal information; in contrast, advertisers in our model interfere with the publisher's learning process in order to do so.

There are a few papers in Computer Science and Operations Research literature that address dynamic learning in repeated auctions. Li et al. (2010) solve for an advertiser's optimal bidding strategy when it is uncertain about its CTR and faces an exogenous distribution of competing bids. Hummel and McAfee (2016) characterize the search engine's optimal bid on behalf of advertisers under uncertain CTRs in a repeated game, and Balseiro and Gur (2017) introduce adaptive bidding strategies for budget-constrained advertisers in repeated auctions of incomplete information.

Closest to our paper within this stream is Iyer et al. (2014), which studies bidding strategies of agents who learn their valuations. Under the assumption that the market size is infinitely large, Iyer et al. (2014) adopt a mean-field approximation to solve for equilibrium strategies. They report a similar finding that in a learning environment, an advertiser's bid consists of the present expected value of winning the ad slot and the "marginal future gain from one additional observation regarding [the advertiser's] valuation." The present paper, however, differs along several important dimensions.

First, since we use performance-based pricing, the learning agent in our model is the publisher, not the advertiser. The publisher receives new information about a new advertiser who wins, and incorporates the information to the rules of the subsequent auctions. Thus, a new advertiser bids strategically not to learn its own type *per se*, but to influence the publisher's learning process. Second, our paper sheds light on a novel incentive for existing advertisers to deter the publisher from learning the new advertiser's type. This is distinct from the idea of advertisers adopting (symmetric) bidding strategies to learn their own types. The discrepancies in the incentives across advertisers that are highlighted in our paper do not emerge in a mean-field equilibrium wherein all agents behave in a symmetric manner. Finally, our paper analyzes a small, stylized market with limited number of participants, which allows us to model fully rational behavior of all players. Our assumption of a small market is motivated by the fact that, due to the fine-grained targeting available in online advertising, most auctions have a small number of participants; as such, advertisers' one-to-one interactions affect their optimal strategies. Papers that employ mean-field equilibrium (e.g., Iyer et al., 2014; Balseiro et al., 2015) abstract away from advertisers' one-to-one interactions, and characterize an approximate equilibrium wherein agents are assumed to be boundedly rational.

2 Model

Our model consists of a publisher and two advertisers, the incumbent and the entrant, indexed by P, I and E, respectively. The publisher sells one ad slot in a second-price auction with reserve price R.[9] Each advertiser has an advertiser-specific CTR — c_I for the incumbent and c_E for the entrant — that represents the average CTR of the advertiser if placed in the ad slot. In other words, when an ad is displayed to a consumer, the consumer clicks on the incumbent's (entrant's) ad with probability c_I (c_E). Parameters c_I and c_E depend on the advertisers' ad copies, as well as the relevance and strength of

[9] We consider a multiple-slot Generalized Second-Price auction in Section 5.

their brands with respect to the publisher's webpage in display advertising, or consumer's search query in search advertising.

In our main model, we assume performance-based pricing, which currently accounts for 62% of the online advertising market in the US,[10] and use pay-per-click pricing terminology.[11] In Section 5.1, we show that, under some assumptions, our findings apply to impression-based pricing as well. We first assume that both advertisers have the same valuation per click, which we normalize to 1. This assumption is not necessary, but simplifies the discussion of advertisers' strategies in Section 3. We relax this assumption in Section 4 when analyzing the publisher's strategy. The incumbent (entrant) submits a bid b_{It} (b_{Et}), where t indexes the game stage. The bids indicate how much the advertisers are willing to pay per click.

In performance-based pricing, publishers take advertisers' expected performance into account when determining payment and allocation. In pay-per-click pricing, publishers compute advertisers' *effective bids* as the product of their submitted bids and the estimated CTRs of their ads.[12] Some publishers may also include other parameters such as landing page experience in the effective bids; however, to focus on the role of CTRs, we only take the submitted bids and the CTRs into account, and assume that the two advertisers are the same along other dimensions that a publisher may consider.[13] Therefore, the effective bids of the incumbent and the entrant at stage t are $c_I b_{It}$ and $c_E b_{Et}$, respectively. The advertiser with the higher effective bid wins the auction, provided its effective bid is greater than or equal to the reserve price, R. The winner pays (per-click) the minimum bid required to win the auction; i.e., if the incumbent wins, it pays $\max[c_E b_{Et}, R]/c_I$ and if the entrant wins, it pays $\max[c_I b_{It}, R]/c_E$.

We assume that c_E is drawn from a differentiable cumulative distribution function (c.d.f.) F_E. Since the incumbent has been advertising with the publisher for an extended period of time, following Edelman et al. (2007) (and many other papers in the literature), we assume that its CTR, c_I, is common knowledge. On the other hand, the entrant's CTR is not known at the time of entry because the entrant has not advertised with the publisher in the past. When the entrant joins, the publisher, the incumbent, and the entrant only know the distribution of the entrant's CTR. [14] We assume that c_I and μ_E, the

[10] http://totalaccess.emarketer.com/chart.aspx?r=219092

[11] Pay-per-click pricing is the most common form of performance-based pricing; nonetheless, our results can be readily applied to other performance-based pricing models such as pay-per-conversion.

[12] For example, see https://www.facebook.com/business/help/430291176997542 and https://searchengineland.com/guide/ppc/how-the-ppc-ad-auction-works.

[13] For a discussion of other parameters in advertisers' effective bids in search advertising see Katona and Zhu (2017).

[14] Note that this model implies two important assumptions on the information structure of the game. First, the assump-

expected value of c_E, are greater than the reserve price, so that the incumbent and the entrant can beat the reserve price in expectation.

Before we proceed, we should elaborate on the meaning of the CTR parameters c_I and c_E. In our model, these parameters represent the *advertiser-specific* CTRs which, as explained above, depend on the advertisers' ad copies and brand strengths among others. Advertiser-specific CTRs are independent of position effects where higher ad slot position increases the ad's click propensity. Indeed, publishers only take into account advertiser-specific CTRs, controlling for position effects, when computing advertisers' effective bids.[15] Position-specific CTRs will be incorporated in the multi-slot extension in Section 5.4.

Next, we describe the timing of the game, which is depicted in Figure 1.

Stage 1: The entrant joins the market. The entrant's CTR is initially unknown, and is therefore set to its expected value μ_E.[16] The incumbent and the entrant simultaneously submit their bids b_{I1} and b_{E1} to the publisher. The incumbent's effective bid is $c_I b_{I1}$ whereas the entrant's is $\mu_E b_{E1}$, since the publisher does not know the entrant's CTR yet. If the incumbent wins, it pays (per-click) $\max[\mu_E b_{E1}, R]/c_I$, and if the entrant wins, it pays $\max[c_I b_{I1}, R]/\mu_E$. If the entrant wins, its CTR becomes known to the publisher by the next stage; otherwise, it remains unknown.

To simplify the analysis, we assume that if the entrant wins a single auction (i.e., the auction in Stage 1), then the publisher learns its CTR. In practice, the entrant would have to win sufficiently many times for the publisher to accurately learn its CTR. Stage 1 in our model corresponds to as many auctions as the entrant needs to win for the publisher to learn its CTR. Furthermore, in practice, learning is continuous and gradual such that the publisher's estimate of the entrant's CTR improves incrementally every time the entrant wins. Our model can be viewed as a discrete approximation of this learning process: the publisher either knows or does not know the entrant's CTR.

Stage 2: The advertisers submit their bids b_{I2} and b_{E2}. The incumbent's effective bid is $c_I b_{I2}$. The entrant's effective bid depends on the outcome of the Stage 1 auction. If the entrant had won in

tion that c_I is common knowledge implies that the entrant and the publisher have the same level of information about the incumbent. Second, we are implicitly assuming that the incumbent and the publisher have the same level of information about the entrant. In practice, it is possible that large publishers such as Google and Facebook can estimate advertisers' CTRs more accurately than other advertisers based on their vast troves of data. We relax both of these assumptions in Section 5.2 and establish the robustness of our results.

[15] `https://support.google.com/google-ads/answer/1659696`

[16] In Google AdWords, new advertisers received an average Quality Score of 6. See `https://searchengineland.com/minimum-quality-score-can-save-money-adwords-226757`. In Section 5.3.2, we consider an extension in which, instead of using μ_E, the publisher strategically sets the entrant's CTR.

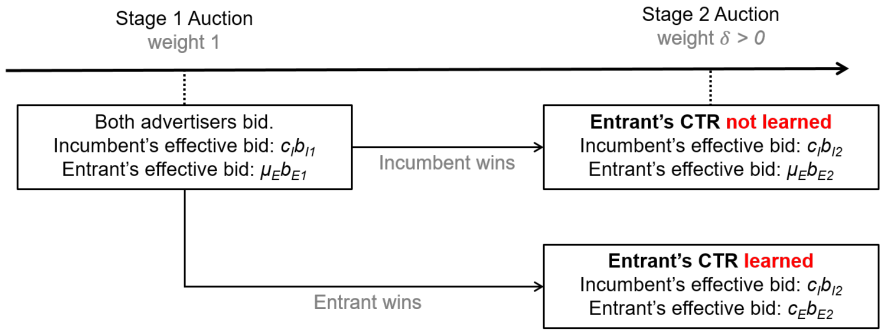


Figure 1: Game Timing

Stage 1, then its CTR becomes known to the publisher by Stage 2, and therefore, its effective bid is $c_E b_{E2}$. Otherwise, as in Stage 1, its CTR is not learned and its effective bid is $\mu_E b_{E2}$.[17]

We capture the relative weight of Stage 2 compared to Stage 1 with parameter $\delta > 0$. Note that since the advertisers' decisions in Stage 1 affects their payoffs in Stage 2, δ affects how the advertisers trade off short-term revenue (in Stage 1) for long-term revenue (in Stage 2).[18]

The incumbent's expected profit is the sum of its first and second stage payoffs. That is, $\mathbb{E}[\pi_I] = \pi_{I1} + \delta\mathbb{E}[\pi_{I2}]$ where π_{I1} denotes the incumbent's first stage payoff, and π_{I2} its second stage payoff contingent on the realization of c_E, over which expectation is taken. Specifically,

$$\pi_{I1} = \begin{cases} c_I\left(1 - \frac{\max[\mu_E b_{E1}, R]}{c_I}\right) & \text{if } c_I b_{I1} \geq \max[\mu_E b_{E1}, R], \\ 0 & \text{otherwise,} \end{cases} \quad \pi_{I2} = \begin{cases} c_I\left(1 - \frac{\max[\tilde{c}_E b_{E2}, R]}{c_I}\right) & \text{if } c_I b_{I2} \geq \max[\tilde{c}_E b_{E2}, R], \\ 0 & \text{otherwise,} \end{cases}$$

where $\tilde{c}_E$ is c_E if c_E is learned (i.e., entrant won in Stage 1 auction), and μ_E otherwise. Similarly, the entrant's expected profit is $\mathbb{E}[\pi_E] = \mathbb{E}[\pi_{E1}] + \delta\mathbb{E}[\pi_{E2}]$, where

$$\pi_{E1} = \begin{cases} c_E\left(1 - \frac{\max[c_I b_{I1}, R]}{\mu_E}\right) & \text{if } \mu_E b_{E1} \geq \max[c_I b_{I1}, R], \\ 0 & \text{otherwise,} \end{cases} \quad \pi_{E2} = \begin{cases} c_E\left(1 - \frac{\max[c_I b_{I2}, R]}{\tilde{c}_E}\right) & \text{if } \tilde{c}_E b_{E2} \geq \max[c_I b_{I2}, R], \\ 0 & \text{otherwise.} \end{cases}$$

[17] If the entrant wins the auction in Stage 1, the publisher learns c_E; however, we do not make any assumptions on whether the incumbent also learns c_E or not. Specifically, as we show in Lemma 1, the incumbent bids truthfully in Stage 2 regardless of the outcome of Stage 1.

[18] One might argue that the publisher eventually learns the entrant's CTR, even if the entrant does not win in Stage 1. For instance, its CTR may be learned if the entrant's ad is displayed on the second page of the search results for a sufficiently long period of time. In this case, we could assume that the game has a Stage 3 in which, regardless of the outcomes of Stages 1-2, c_E becomes learned by the publisher. It is easy to show that both advertisers bid truthfully in Stage 3, and that the existence of Stage 3 does not affect the advertisers' strategies in Stages 1-2. In this model, δ could be interpreted as the length of time required for the publisher to learn the entrant's CTR if the entrant does not win in Stage 1 (compared to when it wins in Stage 1).

Finally, the publisher's expected profit is $\mathbb{E}[\pi_P] = \mathbb{E}[\pi_{P1}] + \delta \mathbb{E}[\pi_{P2}]$, where

$$\pi_{P1} = \begin{cases} \max[\mu_E b_{E1}, R] & \text{if } c_I b_{I1} \geq \max[\mu_E b_{E1}, R], \\ c_E \frac{\max[c_I b_{I1}, R]}{\mu_E} & \text{if } \mu_E b_{E1} > \max[c_I b_{I1}, R], \\ 0 & \text{otherwise,} \end{cases} \quad \pi_{P2} = \begin{cases} \max[\tilde{c}_E b_{E2}, R] & \text{if } c_I b_{I2} \geq \max[\tilde{c}_E b_{E2}, R], \\ c_E \frac{\max[c_I b_{I2}, R]}{\tilde{c}_E} & \tilde{c}_E b_{E2} > \max[c_I b_{I2}, R], \\ 0 & \text{otherwise.} \end{cases}$$

We use subgame perfect Nash equilibrium as the solution concept and solve by backward induction. Finally, to ensure the existence of a weakly dominant strategy for the incumbent, we assume that $c_I + \delta \left((c_I - \mu_E)^+ - \int_0^1 (c_I - \max[c_E, R])^+ \, dF_E \right) \geq R$, for which a sufficient condition is $\delta \leq \frac{1}{f_E(R) + F_E(R)}$.[19] This assumption is only needed to facilitate the exposition in Section 3, and will be dropped in Section 4.

3 Advertisers' Strategies

In this section, we analyze the advertisers' bidding strategies and assume that the publisher's mechanism is exogenous. As a benchmark, in Section 3.1, we analyze the advertisers' strategies in a full information game. Then, in Section 3.2, we study how learning incentives in an incomplete information game affect the advertisers' bidding strategies.

3.1 Full Information Setting

As a benchmark, we first consider the case where the entrant's CTR is common knowledge. This corresponds to what most of the previous theoretical papers in online advertising literature assume. Even though the auction is not a standard second-price auction because advertisers' bids are multiplied by their CTRs, truthful bidding (i.e., bidding the per-click valuation) is still a weakly dominant strategy for both advertisers. The advertisers' equilibrium strategies and their payoffs under full information are summarized in the following proposition.

Proposition 1 (Bids and Payoffs Under Full Information)**.** *Under full information, truthful bidding is a weakly dominant strategy for both advertisers. The payoffs of the incumbent, the entrant, and*

[19] This is not a restrictive assumption; for example, for $F_E(c) = c$, the condition holds for all $\delta > 0$ and $c_I \geq R$. The sufficient condition derives from the fact that $c_I + \delta \left((c_I - \mu_E)^+ - \int_0^1 (c_I - \max[c_E, R])^+ \, dF_E \right)$ is equal to R at $c_I = R$, and then imposing that the former increases in c_I.

the publisher, respectively, are $\pi_I^F = (1+\delta)(c_I - \max[c_E, R])^+$, $\pi_E^F = (1+\delta)(c_E - c_I)^+$, *and* $\pi_P^F = (1+\delta)\max[\min[c_I, c_E], R]$, *where* $x^+ \equiv \max[x, 0]$.

Proposition 1 shows that when the publisher knows the entrant's CTR, both advertisers always bid truthfully. This finding is not new to the literature and is presented here for the sake of completeness. Interestingly, in the next section, we show that truthful bidding is no longer an equilibrium strategy when the publisher does not know the entrant's CTR.

3.2 Incomplete Information Setting

In practice, there is little information regarding the entrant's CTR that is available to the publisher. Therefore, unlike the case for the incumbent's CTR, the advertisers and the publisher have at best only partial information about the entrant's CTR.

We begin our analysis under incomplete information with the second stage bids. We focus on dominant strategy equilibrium where advertisers play weakly dominant strategies. As we show in Lemma 1, Stage 2 auction is straightforward: advertisers bid truthfully. This is because in the last stage there are no strategic considerations of future payoffs; thus, the truthfulness property of standard second-price auctions holds.

Lemma 1 (Bids in Stage 2 Under Incomplete Information)**.** *Regardless of the outcome in Stage 1, bidding truthfully is a weakly dominant strategy for both advertisers in Stage 2.*

In contrast, we find that in Stage 1, the advertisers' bidding strategies are not always truthful. Their bids can be either below or above valuation depending on their expectations of Stage 2 payoffs. The following lemma characterizes the advertisers' first stage equilibrium bids.

Lemma 2 (Bids in Stage 1 Under Incomplete Information)**.** *In Stage 1, it is weakly dominant for the incumbent and the entrant, respectively, to bid*

$$b_{I1}^* = 1 + \frac{\delta}{c_I}\left((c_I - \mu_E)^+ - \int_0^{c_I} c_I - \max[c_E, R]\, dF_E\right), \tag{3.1}$$

$$b_{E1}^* = 1 + \frac{\delta}{\mu_E}\left(\int_{c_I}^1 c_E - c_I\, dF_E - (\mu_E - c_I)^+\right). \tag{3.2}$$

In general, truthful bidding is a weakly dominant strategy in a second-price auction even under incomplete information. Expressions (3.1) and (3.2), however, show that the advertisers' bids are no longer truthful. What drives the change in advertisers' strategies in our model is the advertisers' incentive (or lack thereof) to help the publisher learn the entrant's CTR. The advertisers' Stage 1 bids are shaped by their preference to play a Stage 2 game in which the entrant's CTR is μ_E vs. c_E, where c_E is randomly drawn from F_E. For example, if the entrant's expected payoff in Stage 2 is higher when its CTR is c_E (i.e., its CTR is learned), compared to when it is μ_E (i.e., its CTR is not learned), the entrant would raise its Stage 1 bid.

But does the entrant prefer its CTR to be learned by the publisher? We find the answer to be affirmative. For the entrant, the benefits of revealing its CTR are two-fold. First, it allows the entrant to outrank the incumbent in Stage 2 with some probability even when $\mu_E \le c_I$, a situation in which the entrant would have surely lost in Stage 2 if its CTR was unknown and set to μ_E by the publisher. Second, it provides an opportunity for the entrant to pay lower cost-per-click in the event that its CTR turns out to be high, compared to the case when its CTR is assigned the mean estimate μ_E. Evidently, there is also the risk of its CTR turning out to be low, in which case the entrant would have been better off being assigned μ_E. The reward of a high CTR realization, however, is disproportionately larger than the loss the entrant incurs for a low realization. The reason is that while the gains for the entrant increase proportionally with high realizations of c_E, the loss of a low c_E is bounded from below by zero. Therefore, in expectation, the entrant prefers its CTR to be learned by the publisher.

The following table shows this more formally for the case when $\mu_E > c_I$:

Table 2: Entrant's Stage 2 Profit

Publisher does not know c_E	Publisher knows c_E
$\mathbb{E}[\pi_{E2}] = \int_0^1 c_E(1 - c_I/\mu_E)\, dF_E$	$\mathbb{E}[\pi_{E2}] = \int_0^1 c_E(1 - c_I/c_E)^+\, dF_E$
$= (1 - c_I/\mu_E) \int_0^1 c_E\, dF_E = (1 - c_I/\mu_E)\mu_E$	$= \int_{c_I}^1 c_E(1 - c_I/c_E)\, dF_E$
$= \mu_E - c_I = \int_0^1 c_E - c_I\, dF_E$	$= \int_{c_I}^1 c_E - c_I\, dF_E$

From Table 2, we see that the entrant's Stage 2 profit when the publisher does not know the entrant's CTR (left-hand side) is integrated over negative values as well (in the range $c_E \in (0, c_I)$). This integral value is lower than that when the publisher knows c_E (right-hand side) where only positive values are integrated. In sum, for any entrant CTR distribution F_E, the entrant's Stage 2 profit is higher in expectation if the publisher learns its CTR. Therefore, the entrant bids aggressively in Stage 1 in order to facilitate the publisher's learning process.

The incumbent's bidding strategy is slightly more nuanced: the incumbent underbids for low c_I and overbids for high c_I. Suppose c_E is not learned by the publisher in Stage 2. If c_I is close to μ_E, then the incumbent either loses the Stage 2 auction or, even if it wins the auction, receives a low Stage 2 payoff because the cost-per-click μ_E/c_I is high. In this case, the incumbent is better off shading its Stage 1 bid below valuation, thereby *helping* the entrant win the first stage auction. The intuition is that by facilitating the revelation of the entrant's CTR, the incumbent foregoes its first stage payoff, but creates an opportunity to reap a large second stage payoff in the event c_E turns out to be low. Thus, a weak incumbent has a strategic incentive to underbid.

On the other hand, if c_I is significantly greater than μ_E, then the incumbent's Stage 1 strategy switches from underbidding to overbidding. To illustrate, suppose c_I is high and compare the incumbent's Stage 2 payoff when c_E is concealed vs. revealed. Had c_E been concealed, the incumbent would win in Stage 2 at a low cost-per-click of μ_E/c_I, since $c_I \gg \mu_E$. Conversely, had c_E been revealed, there are two possibilities: if c_E turns out to be low, the incumbent will pay an even lower cost; if c_E turns out to be high, the incumbent will pay a high cost (if not lose the ad position). However, the reward of a low c_E realization is outweighed by the risk of a high c_E realization because the incumbent's potential to reap larger margins for a low c_E realization is limited by the reserve price. Therefore, the incumbent has incentive to conceal c_E when its CTR is high, and thus bids above valuation in Stage 1. This can also be seen from the following expressions of the incumbent's Stage 2 profit when $c_I > \mu_E$:

Table 3: Incumbent's Stage 2 Profit when $c_I > \mu_E$

Publisher does not know c_E	Publisher knows c_E
$\mathbb{E}[\pi_{I2}] = c_I(1 - \mu_E/c_I)$	$\mathbb{E}[\pi_{I2}] = \int_0^1 c_I(1 - \max[c_E, R]/c_I)^+ \, dF_E$
$= c_I - \mu_E$	$= \int_0^R c_I(1 - R/c_I) \, dF_E + \int_R^{c_I} c_I(1 - c_E/c_I) \, dF_E$
$= \int_0^1 c_I - c_E \, dF_E$	$= \int_0^R c_I - R \, dF_E + \int_R^{c_I} c_I - c_E \, dF_E$

From Table 3, we see that the incumbent's Stage 2 profit when the publisher does not know the entrant's CTR (left-hand side) is $c_I - c_E$ integrated over all values of c_E. When c_E is known (right-hand side), for values of $c_E \in (0, R)$, we have $c_I - R$ integrated; since $R > c_E$, the incumbent is better off when the publisher does not know c_E for this integration range. Within the integration range of $c_E \in (R, c_I)$, the expressions on both sides are equal to $c_I - c_E$. Finally, within the range $c_E \in (c_I, 1)$, negative values are integrated on the left-hand side expression whereas the right-hand side expression is zero. For this

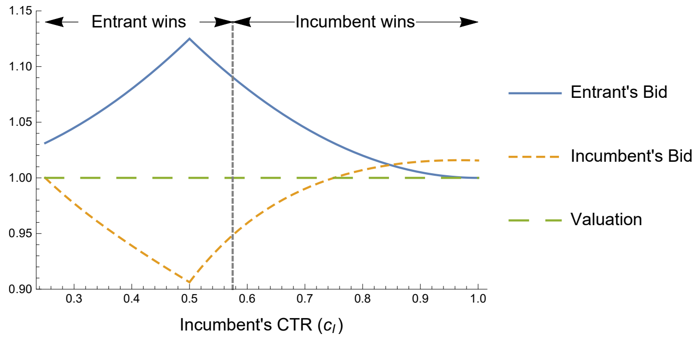


Figure 2: Stage 1 Equilibrium Bids; $R = \frac{1}{4}, \delta = \frac{1}{2}, F_E(c) = c$

integration range, the incumbent is better off when the publisher knows c_E. Overall, the negative effect of learning c_E on the incumbent's profit (which happens for $c_E \in (0, R)$) is constant as c_I increases, but the positive effect (which happens for $c_E \in (c_I, 1)$) shrinks as c_I increases. Therefore, a weak incumbent with low c_I is better off in Stage 2 when c_E is learned, whereas a strong incumbent with high c_I is better off when c_E is not learned. This incentivizes a weak (strong) incumbent to underbid (overbid) in Stage 1. We summarize these results in the following proposition.

Proposition 2 (Advertisers' Strategies in Stage 1 Under Incomplete Information)**.** *In Stage 1, the entrant always bids above its valuation. The incumbent bids below its valuation if c_I is low, and bids above its valuation if c_I is high. See Figure 2.*

The advertisers' bidding behavior outlined in Proposition 2 can also be understood from an *asymmetric learning* perspective. Suppose that the publisher always learns the entrant's CTR in Stage 2, regardless of the Stage 1 outcome. In this hypothetical scenario, the advertisers' Stage 2 payoffs would be independent of the Stage 1 outcome. As a result, neither the incumbent nor the entrant would have incentive to deviate from truthful bidding in Stage 1. In our model, however, the fact that the publisher's learning is asymmetric — that is, learning occurs if only if the entrant wins in Stage 1 — creates an important interdependence between the two sequential auctions. This interdependence, which is depicted in Figure 1, generates strategic incentives for advertisers to deviate from truthful bidding.

Publisher's Revenue

We turn to the implications of learning incentives on the publisher's revenue. Is the publisher unequivocally better off knowing the entrant's CTR? One may conjecture that being more informed about the bidders can only benefit the publisher as it would allow for more efficient ad slot allocation. Surprisingly, we find that this is not always the case. Under certain conditions, *not* knowing the entrant's CTR increases the publisher's revenue.[20]

The intuition revolves around two effects. First, the publisher's ignorance of the entrant's CTR induces the entrant to bid more aggressively in Stage 1. As explained above, the incentive to bid higher arises from the fact that the entrant's expected payoff in Stage 2 is higher if the publisher learns its CTR. This higher bid increases the incumbent's payment if it wins, which results in higher Stage 1 revenue for the publisher.

The second effect is subtler. Consider the publisher's Stage 2 revenue when $c_I > \mu_E$. Recall that the advertisers bid truthfully in Stage 2. If c_E is not known, the publisher's expected revenue in Stage 2 is $(\mu_E/c_I)c_I = \mu_E$. Using the definition of μ_E, this can be rewritten as

$$\int_0^1 c_E \, dF_E. \tag{3.3}$$

If c_E is known, the publisher's Stage 2 revenue depends on the realization of c_E and can be written as

$$\int_0^R R \, dF_E + \int_R^{c_I} c_E \, dF_E + \int_{c_I}^1 c_I \, dF_E. \tag{3.4}$$

Comparing the two integral expressions (3.3) and (3.4), we see that within the integration range $c_E \in (0, R)$, Expression (3.4) is larger; within the range $c_E \in (R, c_I)$, the two expressions are equal, and within the range $c_E \in (c_I, 1)$, Expression (3.3) is larger. Thus, if c_I is not too high, then the publisher's revenue when it does not know c_E (i.e., Expression (3.3)) is larger than when it does (i.e., Expression (3.4)). Intuitively, since the benefit of a high realization of c_E is bounded from above by c_I, i.e., the publisher cannot fully reap the benefits of a high c_E, the publisher's Stage 2 revenue may be higher when c_E is *not* known than when it is. Taken together, the publisher's ignorance of the entrant's CTR can be blissful for moderate values of c_I. This result is formalized in the following proposition.

Proposition 3 (Publisher Revenue: Ignorance is Bliss)**.** *The publisher's revenue is higher not knowing the entrant's CTR than knowing it if and only if (i)* $\underline{c} < c_I < \overline{c}$*, or (ii)* $c_I \leq \mu_E$ *and* $\delta < 1$*, where* $\underline{c}$ *and* $\overline{c}$ *are defined in the appendix.*

[20]To be more precise, the common knowledge that the publisher does not know the entrant's CTR may increase its revenue.

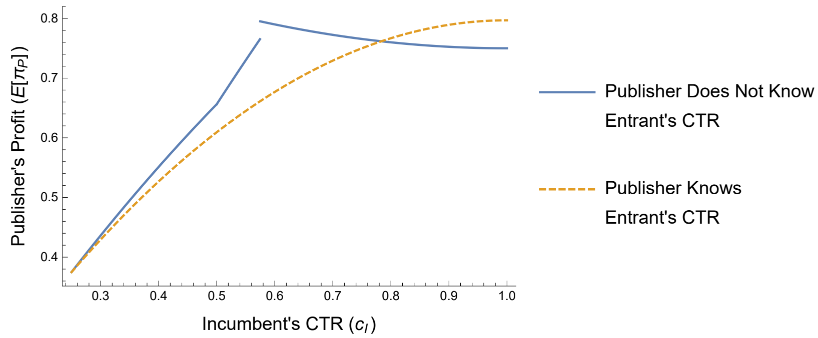


Figure 3: Publisher Revenue; $R = \frac{1}{4}, \delta = \frac{1}{2}, F_E(c) = c$

Proposition 3 suggests that publishers do not always have to be concerned about not knowing the new advertisers' types. In fact, not knowing the new advertisers' CTRs can sometimes increase the publisher's revenue because ignorance induces advertisers to bid aggressively. However, Proposition 3 also reveals conditions under which the publisher's ignorance can be a curse. For instance, if the incumbent is strong (e.g., high c_I in Figure 3), then not knowing the entrant's CTR decreases the publisher's revenue. This is because, when c_I is sufficiently high, the entrant, who is the "price setter" in the auction, bids less aggressively. Furthermore, when c_I is high, the publisher does not learn the entrant's CTR in equilibrium due to the incumbent's aggressive bidding. As a result, it suffers from suboptimal allocation of the ad slot (i.e., missing out on a potentially high c_E).

Given that the publisher may incur a revenue loss for not knowing c_E, one may wonder what strategies a publisher can deploy to mitigate this loss. In the next section, we characterize the publisher's optimal strategy in a learning environment. We show that, in the presence of learning incentives, it is optimal for the publisher to favor the entrant in Stage 1 in order to increase the probability of the entrant's winning.

4 Publisher's Strategy

In the previous section, we assumed that advertisers have the same, commonly known valuation for the ad slot. Moreover, we focused primarily on the advertisers' strategies, with the publisher passively implementing an exogenously fixed auction mechanism. In this section, we analyze a setting where advertisers have stochastic, private valuations and, more importantly, the publisher optimally chooses the mechanism that maximizes its profit.[21] We show that, in the presence of learning incentives, the publisher can achieve the optimal revenue using a variation of the standard second-price auction with personalized (advertiser-specific) reserve prices. Additionally, the learning incentives induce the publisher to favor the entrant in Stage 1.

4.1 Optimal Mechanism

Suppose advertiser j's per-click valuation, v_j, is drawn from a c.d.f. G_j with support $[0, \overline{v}_j]$ and is private information, for $j \in \{I, E\}$. Following the literature on auction theory (see Krishna, 2010), we impose the following assumption on G_j.

Assumption 1 (Increasing Hazard Rate)**.** *Let g_j denote the density of G_j. The hazard rate function $\frac{g_j(x)}{1-G_j(x)}$ is increasing in x for $j \in \{I, E\}$.*[22]

Prior to Stage 1, the publisher sets the ad auction rules. In particular, it decides the allocation rule (who wins the ad slot), and the payment rule (how much each bidder pays). The rest of the game proceeds the same as in Section 3. The following lemma characterizes the publisher's optimal mechanism.

Lemma 3 (Publisher's Optimal Mechanism)**.** *The publisher's optimal mechanism is as follows.*

Stage 1: Compute the incumbent's and entrant's virtual bids, respectively, as

$$\psi_{I1}(b_{I1}) = c_I \left(b_{I1} - \frac{1 - H_I(b_{I1})}{h_I(b_{I1})} \right) \text{ and } \psi_{E1}(b_{E1}) = \mu_E \left(b_{E1} - \frac{1 - H_E(b_{E1})}{h_E(b_{E1})} \right) + \delta\Delta_P, \quad (4.1)$$

and set the virtual reserve price to $\delta\Delta_I$.

Stage 2: Compute advertiser j's virtual bid as

$$\psi_{j2}(b_{j2}) = c_j \left(b_{j2} - \frac{1 - G_j(b_{j2})}{g_j(b_{j2})} \right) \text{ for } j \in \{I, E\},^{23} \quad (4.2)$$

[21] In order to characterize the optimal mechanism, we have to assume stochastic private valuations for the advertisers; otherwise, the publisher's optimal strategy is to set the reserve price of Stage 2 to 1, leaving no surplus for the advertisers. Stochastic private valuation is a standard assumption in mechanism design literature; e.g., see Myerson (1981) for a general setting, and Edelman and Schwarz (2010) for the context of online advertising.

[22] Assumption 1 greatly facilitates the derivation of the optimal mechanism. A large class of distributions satisfy this property; e.g., exponential, Weibull, modified extreme value, Gamma (with parameters $\alpha > 1, \lambda > 0$), and truncated normal (with "commonly accepted [parameters]"). See Barlow and Proschan (1965) and Brusset (2009) for details.

[23] We are slightly abusing notation: "c_E" in Stage 2 is $\tilde{c}_E$, which is c_E if c_E is learned, and μ_E otherwise.

and set the virtual reserve price to 0,

where $H_j(b_{j1}) = G_j\left(b_{j1} - \frac{\delta\Delta_j}{c_j}\right)$, $\Delta_I = \pi_{I2}(\mu_E) - \int_0^1 \pi_{I2}(c_E)\, dF_E$, $\Delta_E = \int_0^1 \pi_{E2}(c_E)\, dF_E - \pi_{E2}(\mu_E)$, $\Delta_P = \int_0^1 \pi_{P2}(c_E)\, dF_E - \pi_{P2}(\mu_E)$, *and* $\pi_{j2}(c'_E)$ *denotes the Stage 2 profit of player* $j \in \{I, E, P\}$ *under the optimal Stage 2 mechanism when the publisher assigns entrant's CTR as* c'_E.

Allocate the ad slot to the advertiser with highest virtual bid, provided it exceeds the virtual reserve price. Payment (per-click) is equal to the minimum bid required for the winning advertiser to win.

The details of the proof are provided in the appendix. We briefly discuss here the intuition behind the optimal mechanism. Variables Δ_j, $j \in \{I, E, P\}$, capture the difference in a full-information Stage 2 vs. an incomplete-information Stage 2 in the players' payoffs; i.e., Δ_E measures the additional Stage 2 payoff the entrant gains from having its CTR learned by the publisher; Δ_P measures the additional Stage 2 payoff the publisher gains from learning the entrant's CTR; and Δ_I represents the additional Stage 2 payoff the incumbent gains if the entrant's CTR is *not* learned. Distributions H_j, $j \in \{I, E\}$, are similar to advertisers' valuation distributions G_j, except that they are shifted to account for the advertisers' incentives to have the entrant's CTR learned or not learned.

The derivation of the optimal mechanism closely follows Myerson (1981). The optimal mechanism in Stage 2, where learning incentives are absent, is a direct application Myerson's lemma. Intuitively, the virtual bid transformation amounts to sorting advertisers based on the marginal revenue they bring to the publisher (Krishna, 2010). Thus, allocating the ad slot to the advertiser with the highest virtual bid maximizes the publisher's profit.

In Stage 1, the presence of learning incentives (for both the advertisers and the publisher) distorts the advertiser's virtual bids compared to the standard format in Myerson (1981). Specifically, we see from (4.1) that the publisher additively inflates the entrant's virtual bid by $\delta\Delta_P$. This term represents the additional Stage 2 payoff the publisher gains from learning the entrant's CTR and is proven to be always positive.[24] Intuitively, since the publisher can only learn the entrant's CTR if the entrant wins in Stage 1, the publisher has an incentive to help the entrant win. The publisher accomplishes this by increasing the entrant's virtual bid in Stage 1.[25]

[24] To see that Δ_P is positive, it suffices to show $\pi_{P2}(c_E) = \iint \max\left[\psi_{I2}(x_{I2}|c_I), \psi_{E2}(x_{E2}|c_E)\right]^+ \, d\mathbf{G}$ is convex in c_E. The integrand is convex in c_E because it is the maximum of $\psi_{I2}(x_{I2}|c_I)^+$, which is independent of c_E, and $\psi_{E2}(x_{E2}|c_E)$ which is a linear function of c_E. And since any linear combination with positive weights of convex of functions is also convex, we conclude $\pi_{P2}(c_E)$ is convex in c_E.

[25] It can be easily verified that the Stage 1 virtual bids in (4.1) reduce to the standard format (Myerson, 1981) when the

Lemma 3 also sheds light on the nature of the optimal virtual bids. For example, if the advertisers' valuations are uniformly distributed, then it is optimal for the publisher to compute virtual bids by multiplying the advertisers' bids with their expected CTRs (modulo an additive term). This implies that publishers with diffuse priors about advertisers' valuations can achieve near-optimal revenues by ranking advertisers based on CTR $\times$ bid. Moreover, the fact that the CTR-multiplier formula is also used in Stage 1 in the presence of learning dynamics attests to the robustness of this particular virtual bid format.

Next, we discuss the advertisers' bidding strategies under the optimal mechanism. Interestingly, we find that the insights from Section 3 regarding bid adjustments carry over to the optimal mechanism setting. As shown in Figure 4, the entrant overbids in Stage 1. Its motivation closely mirrors that of Section 3: its expected payoff in Stage 2 is greater if its CTR is learned by the publisher because the downside risk of a low c_E draw is bounded.

A weak incumbent bids below its valuation and helps the entrant reveal its CTR. In contrast to Section 3, however, the heterogeneity in advertisers' valuations necessitates an additional condition for this result to hold. Namely, the valuation distributions G_I and G_E must be such that the weak incumbent's probability of winning in Stage 2 decreases sufficiently slowly in c_E. Roughly, this is equivalent to the incumbent's valuation distribution being more concentrated around higher values than is the entrant's valuation distribution. For then, even if the entrant's CTR turns out to be high in Stage 2, the incumbent, whose valuation is more heavily concentrated on higher values, would still have a considerable chance of winning. This condition ensures the weak incumbent, who effectively helps the entrant win in Stage 1, feels adequately "insured" against the risk of a high c_E draw in Stage 2. The weak incumbent will then forego its Stage 1 profit and help reveal the entrant's CTR, as it creates an opportunity to earn higher profits against an entrant with a low CTR draw.

Finally, a strong incumbent may overbid under the optimal mechanism (see Figure 4). Again, the intuition mirrors that from Section 3; however, the added necessary condition is that the incumbent's probability of winning in Stage 2 decrease steeply in the entrant's CTR draw. In this case, the incumbent deems the risk of revealing the entrant's CTR in Stage 2 too high. Therefore, it bids aggressively in Stage 1 and deters the publisher from learning the entrant's CTR. We summarize these findings in the following proposition.

learning dynamics are muted (e.g., $\delta = 0$).

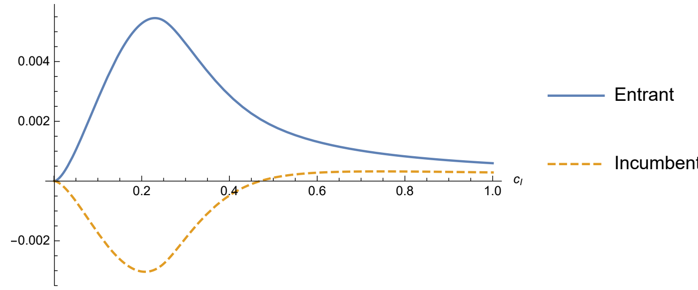


Figure 4: Advertisers' Bid Adjustments Relative to Valuation; $G_I(x_I) = \left(\frac{x_I}{2}\right)^4, G_E(x_E) = x_E$

Proposition 4 (Advertisers' Strategies in Stage 1 Under Optimal Mechanism)**.** *Suppose the publisher implements the optimal mechanism characterized in Lemma 3. In Stage 1, the entrant always bids above its valuation. The incumbent bids below its valuation if c_I is low and the valuation distributions G_I and G_E are such that the probability of the incumbent winning in Stage 2 decreases sufficiently slowly in the entrant's CTR draw. The incumbent bids above its valuation if c_I is high and G_I and G_E are such that the incumbent's probability of winning in Stage 2 decreases steeply in the entrant's CTR draw. The exact conditions are provided in the proof in Section A.7.*

4.2 Optimal Reserve Prices

In this section, we delve deeper into a particular aspect of the optimal mechanism, the (effective) reserve prices. We examine how the learning incentives affect the publisher's optimal reserve prices in Stage 1. Before we proceed, we should clarify two distinct units of reserve prices. *Virtual reserve price* is defined in the virtual bids space, and measures the *minimum virtual bid* required for an advertiser to participate. *Effective reserve price* is defined in the submitted bids space, and refers to the *minimum submitted bid* required for a particular advertiser to participate (Ostrovsky and Schwarz, 2016). To illustrate, suppose the publisher sets a virtual reserve price \$0.15 and an advertiser bids b. Suppose that the advertiser's virtual bid is set to $0.1 \times b$ by the publisher. The advertiser will be considered in the auction if its virtual bid $0.1 \times b$ is greater than or equal to \$0.15. Equivalently, the advertiser will be considered if its *submitted* bid b is greater than or equal to the *effective reserve price* $\$0.15/0.1 = \$1.5$.

Although the optimal mechanism uses the same virtual reserve price for all advertisers, because it applies different virtual bid transformations, advertisers experience different effective reserve prices. In the following, we consider the effective reserve price as it is a more intuitive concept to discuss

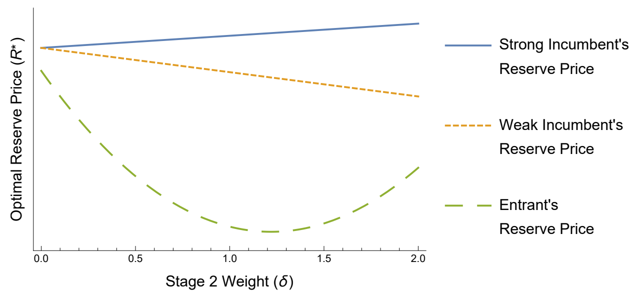


Figure 5: Optimal Reserve Prices (not to scale); $G_I(v_I) = \left(\frac{v_I}{2}\right)^3, G_E(v_E) = \left(\frac{4v_E}{3}\right)^3, F_E(c) = c$

advertisers' payments.

Our analysis shows that the optimal reserve price depends crucially on two countercurrent forces. On the one hand, the *entrant's overbidding incentive* exerts an upward force on the entrant's reserve price. That is, the higher is the value that the entrant gains from the publisher learning its CTR, the higher it bids in Stage 1. The publisher can, thus, extract more from the entrant by setting a higher reserve price. Therefore, the reserve price increases with the entrant's overbidding incentive. The converse is true for the incumbent: the reserve price for the incumbent increases with the incumbent's value of *deterring* the publisher from learning the entrant's CTR.

On the other hand, the *publisher's learning incentive* pushes the entrant's reserve price downward. The higher is the value for the publisher if it learns the entrant's CTR, the greater its incentive to help the entrant win in Stage 1. This is accomplished by lowering the entrant's reserve price. Therefore, the reserve price decreases with the publisher's learning incentive.

Figure 5 illustrates the dynamics of each advertiser's optimal reserve price for particular valuation distributions G_I and G_E. Interestingly, the entrant's reserve price can be non-monotonic in the Stage 2 weight parameter, δ. When δ is small, the publisher learning incentive dominates, and as δ increases, the publisher lowers the reserve price to facilitate learning the entrant's CTR. When δ is large, however, the entrant overbidding incentive dominates. Here, as δ increases, the publisher increases the reserve price to extract more surplus from the entrant.

So far, we have discussed the two countercurrent forces that induce the publisher to increase/decrease the entrant's reserve price in a learning environment. But what is the net effect of these forces on the entrant's reserve price? We conclude this section by presenting the conditions under which the publisher sets a lower reserve price for the entrant when it does not know the entrant's CTR than under full information. While a closed-form characterization of the optimal reserve price with respect to δ is intractable, we can analytically delineate the conditions for when the publisher sets a lower reserve price. In Proposition 5, we show that, for any δ, the publisher sets a lower reserve price for the entrant in a learning environment (compared to full information) if and only if the increment in its Stage 2 profit from learning the entrant's CTR is sufficiently high.

Proposition 5. *The publisher sets a lower reserve price for the entrant when the publisher does not know the entrant's CTR than when it does if and only if $\Delta_P > \Delta_I + \frac{\mu_E \rho}{\delta}$ (where Δ_j is as defined in Lemma 3 and $\rho > 0$ is as defined in the proof); i.e., the publisher's gain in Stage 2 from learning the entrant's CTR is sufficiently high.*

The results of Lemma 3 and Proposition 5 show that when the publisher's learning incentives are sufficiently strong, it is optimal for the publisher to *favor* the entrant. Favoring the entrant can be implemented by increasing the entrant's virtual bid as in Lemma 3, or decreasing the entrant's reserve price as in Proposition 5. In Section 5.3, we show that other mechanisms that favor the entrant can create a similar effect. For example, giving free advertising credit to new advertisers, or artificially inflating the "estimated" CTR of new advertisers can increase the publisher's revenue.

5 Extensions

We present four extensions of the main model to assess the robustness of our results. In Section 5.1, we show that our results from the main model continue to hold under impression-based pricing. In the main model, we assumed that the advertisers and the publisher have the same level of information about the CTRs; in Section 5.2, we relax this assumption to establish the robustness of our results. In Section 5.3, we explore other mechanisms that help the publisher increase its revenue in a learning environment. We show that offering free ad credit to new advertisers, or artificially inflating the "estimated" CTR of new advertisers can increase the publisher's revenue. Finally, in Section 5.4, we turn to the context of search advertising and discuss how our results change when there are multiple ad slots.

5.1 Impression-based Pricing

Suppose the publisher sells a single ad slot through a cost-per-impression (CPM) auction instead of cost-per-click. Consistent with practice, we assume that the publisher receives the advertisers' per-impression bids and assigns the slot to the highest bidder.[26] The winning bidder pays the minimum bid required to win the auction.

To model a CPM auction, it is important to recognize that the relevant performance metric for advertisers is the consumers' "estimated action rates" per ad impression. Consumer actions can range from clicking a link to watching a video longer than a certain amount of time, or completing a certain task on the advertiser's website. For $j \in \{I, E\}$, let $a_j \in [0,1]$ denote the probability of action of a consumer conditional on viewing advertiser j's ad. We normalize the value of a consumer's action to 1 for both advertisers. Put together, a single impression is worth a_j to advertiser j.

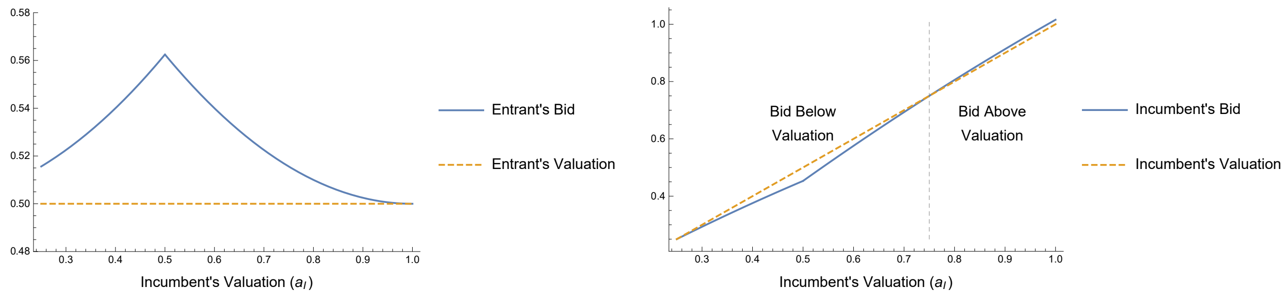
In the spirit of the main model, assume that the action probability associated with the entrant's ad, a_E, is known only up to its c.d.f. $\tilde{F}$ with mean $\tilde{a}_E$, while the incumbent's action probability, a_I, is common knowledge.[27] The true value of a_E can only be learned if the entrant wins the first stage auction; only then can the entrant accurately assess the likelihood of a consumer responding to its ad with some pre-defined action. Similar to Section 3, we assume that the reserve price is exogenously set at R, and is less than a_I and $\tilde{a}_E$. In addition, as in Section 3, we assume $a_I + \delta\left((a_I - \tilde{a}_E)^+ - \int_0^1 (a_I - \max[a_E, R])^+ \, d\tilde{F}\right) \geq R$ to ensure the existence of weakly dominant strategies. Analyzing the advertisers' strategies yields the following proposition, which echoes the results from the main model.

Proposition 6 (Advertisers' Strategies in Stage 1 Under CPM)**.** *In Stage 1, the entrant always bids above its valuation. The incumbent bids below its valuation if a_I, the action probability associated with its ad, is low, and bids above valuation if a_I is high. See Figure 6.*

Proposition 6 shows that advertisers' strategies under impression-based pricing are similar to those under performance-based pricing. However, there is an important difference in the advertisers' incentives between the two pricing models. Under performance-based pricing, the entrant does not care about

[26] This contrasts with rankings based on effective bids in CPC auctions, where publishers multiply the advertiser's bid with its Quality Score.

[27] For our analysis, we only need the incumbent's WTP to be common knowledge; this is a standard assumption in papers with a full-information setting in online advertising, e.g.,Edelman et al. (2007).



(a) Entrant's Bidding Strategy *(b) Incumbent's Bidding Strategy*

Figure 6: CPM Bidding Strategies; $R = \frac{1}{4}, \delta = \frac{1}{2}, \tilde{F}(a) = a$

learning the CTR itself; it overbids so that the *publisher* learns the CTR. In contrast, under impression-based pricing, since the CTR determines the entrant's valuation per impression, the entrant overbids so that the entrant itself learns its CTR. Similarly, under performance-based pricing, the incumbent adjusts its bid to affect whether the publisher learns the entrant's CTR or not, whereas under impression-based pricing, the incumbent wants to affect whether the entrant learns its own CTR or not. Despite discrepancies in the advertisers' incentives across these two pricing models, the mathematical expressions that capture the advertisers' payoffs are relatively similar: in performance-based pricing, the publisher includes the advertiser's CTR in the effective bid, whereas in impression-based pricing the advertiser includes the CTR in its submitted bid. As such, we obtain the same strategic behavior under both pricing models.

5.2 Information Asymmetry

In this section, we test the robustness of our results when the information symmetry assumption is relaxed. We examine two distinct cases. In Section 5.2.1, we demonstrate that the qualitative insights of the main model carry over when the entrant is at an informational disadvantage; i.e., the entrant only knows the incumbent's CTR up to some distribution, while the incumbent and the publisher know the incumbent's true CTR. In Section 5.2.2, we replicate the incumbent's bidding pattern from the main model in a setting where the incumbent only knows the distribution of the entrant's CTR up to a distribution, while the entrant and the publisher know the true distribution of c_E.

5.2.1 Entrant Does Not Know True c_I

In the main model, we assumed that the entrant, incumbent and the publisher possessed the same level of information regarding the incumbent's CTR. In reality, it could be argued that the entrant may not be as knowledgeable about c_I as the incumbent and the publisher. In this section, we analyze a model that captures this information asymmetry more realistically. Our objective is to show that the the entrant's overbidding behavior is robust to the setting where the entrant is less informed about c_I than the incumbent and the publisher.

To that end, suppose that the entrant does not know the true value of c_I, but knows that it follows some distribution F_I over the support $[R, 1]$. On the other hand, the publisher and the incumbent both know the true c_I. We find that the entrant's overbidding pattern is robust to this information setting. The following proposition summarizes the finding.

Proposition 7. *Suppose the entrant does not know the true CTR of the incumbent, but only knows its distribution. In Stage 1, the entrant always bids above its valuation.*

We know from Section 3 that the entrant overbids for any given value of c_I. Intuitively, when the entrant only knows the distribution of c_I, it integrates over all possible values of c_I to calculate its optimal bid. Since the optimal strategy for any value of c_I is to overbid, the optimal strategy when c_I is not known (i.e., the outcome of the integration) is still to overbid. This is formalized in the proof of Proposition 7.

5.2.2 Incumbent Does Not Know True Distribution of c_E

Another important case to consider is when the incumbent, as opposed to the entrant, is at an informational disadvantage. How would the incumbent having less information about c_E than the entrant and the publisher impact its bidding strategy? To address this question, we allow for the possibility that the incumbent does not know the true distribution of c_E; i.e., the incumbent strategizes based on some prior belief over a range of possible distributions of c_E. The publisher and the entrant, on the other hand, know the true distribution of c_E.

Suppose the incumbent believes that the true distribution of c_E is F_x for some $x \in \mathcal{X}$. Let $P(x)$ denote the c.d.f. of the incumbent's prior over the class of distributions $\{F_x\}_{x \in \mathcal{X}}$, and μ_x the mean of F_x. In addition, as in Section 3, assume that the condition for the existence of weakly dominant strategies holds: $c_I + \delta \int_{x \in \mathcal{X}} \left((c_I - \max[\mu_x, R])^+ - \int_0^1 (c_I - \max[c_E, R])^+ \, dF_x \right) dP(x) \geq R$. Then there exists CTR thresholds such that a weak incumbent (i.e., with a low CTR) bids below its valuation, whereas a strong incumbent (i.e., with a high CTR) bids above it. We state this as a proposition.

Proposition 8. *Suppose the incumbent does not know the true distribution of the entrant's CTR. There exists a pair of thresholds* $(\underline{c}', \overline{c}')$ *such that the incumbent bids below its valuation if* $c_I < \underline{c}'$*, and bids above valuation if* $c_I > \overline{c}'$*.*

The intuition for the result of Proposition 8 is similar to that of Proposition 7. For any distribution F_x, we know from Section 3 that a weak incumbent underbids and a strong incumbent overbids. When the incumbent does not know the entrant's distribution, it integrates over all possible distributions to calculate its optimal bid; however, the same pattern continues to exist. When the incumbent is sufficiently weak it overbids and when it is sufficiently strong it overbids. This is formalized in the proof of Proposition 8.

5.3 Other Mechanisms

In this section, we discuss two other mechanisms that can help publishers increase their revenue in a learning environment.

5.3.1 Free Advertising Credit for New Advertisers

Publishers run promotions that provide free ad credit to new advertisers. For example, Google offers ad coupons to new advertisers worth up to $75 which can be redeemed within 30 days of spending the first $25 in advertising.[28] Similarly, Facebook sends promotional codes to new advertisers that a have sufficiently high user engagement on their pages. In this section, we study the implications of offering ad credit on the advertisers' bidding strategies and the publisher's profit.

Suppose the publisher sets the ad credit $\alpha \geq 0$ prior to Stage 1, and then the incumbent's CTR is drawn from c.d.f. F_I with support $[R, 1]$.[29] The advertisers observe α and the rest of the game proceeds

[28] `https://www.google.com/ads/adwords-coupon.html`

[29] We assume α is set before c_I is realized because publishers use the same amount of ad credit across many keywords for which incumbents have different CTRs. That is, in practice, α is not a function of c_I. In addition, note that if α is decided after c_I is realized, the truthfulness nature of these second-price auctions will break down. This is because the incumbent

identically as in the main model. The effect of the publisher's ad credit is to transfer free ad credit α to the entrant if it wins in Stage 1. Thus, the entrant's Stage 1 payoff when the publisher offers ad credit α is $c_E\left(1-\frac{\max[c_I b_{I1},R]}{\mu_E}\right)+\alpha$ if it wins, and 0 otherwise. Next, we present the advertisers' bidding strategies when the publisher offers free ad credit.

Proposition 9 (Advertisers' Strategies with Ad Credit)**.** *If the publisher offers ad credit $\alpha \geq 0$ to the entrant, then compared to the benchmark bids (3.1) and (3.2), the incumbent's first stage bid remains unchanged, whereas the entrant's bid increases by $\frac{\alpha}{\mu_E}$ to*

$$b^*_{E1}(\alpha) = 1 + \frac{\alpha}{\mu_E} + \frac{\delta}{\mu_E}\left(\int_{c_I}^{1} c_E - c_I\, dF_E - (\mu_E - c_I)^+\right). \tag{5.1}$$

The intuition behind advertisers' bidding strategies is straightforward: the incumbent's bidding strategy does not change because α does not affect its underlying payment mechanism. However, the ad credit increases the entrant's payoff when it wins in Stage 1, and thus incentivizes the entrant to bid more aggressively in the first stage.

We turn to the impact of ad credit on the publisher's revenue. Given the first stage bids of the incumbent and the entrant in (3.1) and (5.1), respectively, the publisher's expected revenue as a function of ad credit α is $\mathbb{E}[\pi_P(\alpha)] = \int_R^1 \Pi_P(\alpha, c_I)\, dF_I$ where

$$\Pi_P(\alpha, c_I) = \begin{cases} \mu_E b^*_{E1}(\alpha) + \delta \min[c_I, \mu_E] & \text{if } c_I b^*_{I1} \geq \mu_E b^*_{E1}(\alpha), \\ \max[c_I b^*_{I1}, R] - \alpha + \delta\left(\int_0^{c_I} \max[c_E, R]\, dF_E + (1 - F_E(c_I))c_I\right) & \text{if } c_I b^*_{I1} < \mu_E b^*_{E1}(\alpha). \end{cases} \tag{5.2}$$

Expression (5.2) reveals the three forces created by the ad credit α. The first is the cost of α that is transferred from the publisher to the entrant when it wins; this has a negative effect on the publisher's revenue, and is represented by $-\alpha$ in the second case of Expression (5.2). The two other forces have a positive effect on the publisher's revenue, and are more nuanced; we discuss each of these in turn.

Recall that the entrant's bid increases in proportion to the ad credit α (see Proposition 9). This implies that the incumbent's payment upon winning increases with α. The publisher can thus extract additional surplus from the incumbent by inflating its payment. We call this the *extraction effect.*

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would anticipate the publisher to set α high enough to extract all surplus from the incumbent's bid, creating incentives for the incumbent to shade its bid. Finally, the assumption that the ad credit is only available in Stage 1 reflects the fact that these promotions typically expire after a short period of time.

The last effect of ad credit concerns the change in the publisher's Stage 2 payoff. To illustrate, suppose the incumbent's CTR is high. In this case, knowing the entrant's CTR leads to a higher publisher revenue than that under ignorance (see Proposition 3). This is due to the more efficient allocation of the ad slot as well as a higher expected payment of the winner. Since offering ad credit helps the entrant win, thereby facilitating the publisher learning its CTR, it could increase the publisher's Stage 2 revenue. We call this the *learning effect*.

These three effects summarize all the pros and cons of offering ad credit in our model.[30] Due to space considerations, we relegate the extended discussion to Section OA1 of the online appendix. In the discussion, we (i) delineate the condition under which offering ad credit is profitable for the publisher, (ii) characterize the extraction effect and learning effect more formally, and (iii) discuss the optimal ad credit level that maximizes the publisher's profit.

5.3.2 Inflating the Bid Multiplier

In this section, we analyze how the publisher's profit would change if, instead of offering ad credit (Section 5.3.1), the publisher artificially inflated the entrant's effective bid by a multiplier $\beta \geq 1$. To begin, suppose that the publisher applies a boosting multiplier β such that for any bid b_E of the entrant, the entrant's Stage 1 effective bid $\mu_E b_E$ increases to $\beta \times \mu_E b_E$. The rest of the game proceeds as in the main model.

We find that the two policies — offering ad credit and multiplicatively boosting the effective bid — have the same qualitative implications for the publisher's profit. The intuition is as follows. In the case of ad credit, the entrant increases its own effective bid by bidding high in anticipation of the ad credit, whereas in the case of boosting multiplier, the publisher increases the effective bid on behalf of the entrant. Thus, the resultant effective bids across the two policies are the same, and the players' payoffs are identical up to a constant. We formalize this finding in the following proposition.

Proposition 10. *The multiplicative boosting policy is isomorphic to the free ad credit policy, in the sense that the publisher can replicate (up to a constant) its profit from one policy using the other.*

[30]Note that we are not discussing the *customer acquisition effect* of offering free ad credit. Promotional incentives for attracting new customers have been extensively studied in the literature (e.g., Jedidi et al., 1999; Nijs et al., 2001; van Heerde et al., 2003). Instead, we focus on the extraction and learning effects of ad credit that are new to the literature. We show that even if the free ad credit does not attract new advertisers, the publisher may still benefit from offering it because of these two positive effects.

In summary, we find that other mechanisms such as offering free ad credit or inflating the entrant's effective bid can increase the publisher's revenue as they facilitate learning by favoring the entrant. These mechanisms, however, are inefficient in the sense that they do always guarantee the publisher the optimal profit. To see this, note that the optimal mechanism sometimes lowers the payment for both the entrant and the incumbent at the same time (e.g., when the incumbent's CTR is low). Such an outcome cannot be produced by offering ad credit or inflating the entrant's bid because these instruments unilaterally benefit the entrant at the expense of the incumbent.

Another reason why these alternative mechanisms do not always yield the optimal profit pertains to the virtual bid transformations. Offering ad credit and inflating the entrant's bid do not take into account the advertisers' valuation distributions in the allocation rule. Recall that the optimal mechanism computes the marginal revenue each advertiser generates based on the advertisers' valuation distributions and then allocates the ad slot accordingly. Such efficient allocation cannot always be attained with artificial adjustment of the entrant's effective bid, especially when the valuation distributions are non-uniform.

5.4 Multiple Advertising Slots

Another assumption in the main model is that the publisher offers a single ad slot. In search advertising, however, search engines typically sell more than one ad slot, which are allocated via the Generalized Second-Price (GSP) auction. In this section, we test whether the main insights derived from the base model carry over to the multiple-slot GSP setting.

We consider a two-slot, three-player game where two incumbents face an entry from a new advertiser. To simplify the analysis, we assume that the existing advertisers also learn c_E if the entrant wins in Stage 1.[31] As in Section 3, all advertisers share a common per-click valuation of 1, and the reserve price R is less than μ_E. We normalize the position-specific CTR of the first ad slot to 1 and denote that of the second slot as $\theta \in (0, 1)$. We index by i and I the incumbent with the lower and higher CTR, respectively (i.e., $c_i < c_I$), and normalize c_I to 1.[32] Since a weakly dominant strategy equilibrium no

[31] This could be justified by the fact that advertisers can estimate the effective bid of the advertisers below them by observing the amount they are charged.

[32] The normalization can also be interpreted as assuming that the CTR of the average entrant does not exceed that of the strong incumbent. This assumption simplifies expressions but is not necessary. The analysis without this assumption is provided in the appendix.

longer exists, we use the lowest-revenue envy-free (LREF) Nash equilibrium (Edelman et al., 2007) for equilibrium selection.

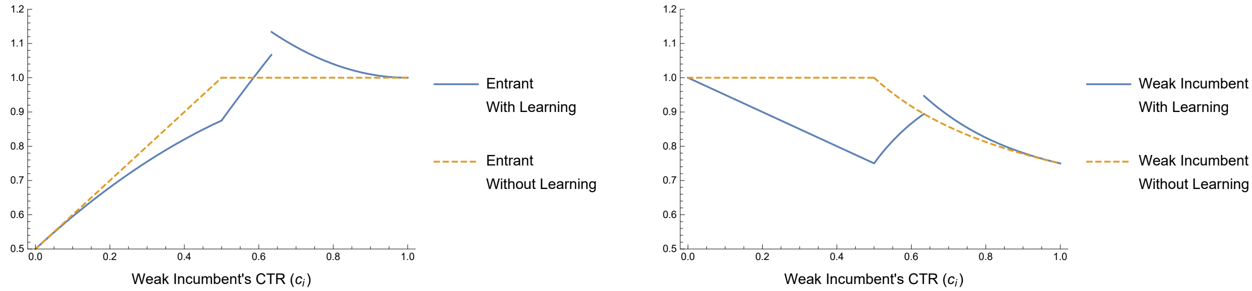
Our analysis shows that the main results are robust to multiple-slot settings under GSP auction. In particular, the entrant's overbidding strategy carries over, but with the caveat that the reserve price has to be sufficiently high. Moreover, the findings that (i) a weak incumbent prefers to reveal the entrant's CTR, and that (ii) a strong incumbent has incentives to mask the entrant's CTR are preserved in the multiple-slot extension. The next lemma summarizes the advertisers' incentives in Stage 1.

Lemma 4 (Multiple-Slot GSP Auction)**.** *The entrant and the incumbent with the lower CTR are always better off in Stage 2 if the entrant's CTR is learned. The incumbent with the higher CTR is better off in Stage 2 if the entrant's CTR is learned if and only if (i) $\mu_E < c_i$ and $\theta > \hat{\theta}$; or (ii) $c_i \leq \mu_E$ and $\theta > \frac{1}{2}$, where $\hat{\theta}$ is defined in the appendix.*

Lemma 4 shows that if the second ad slot generates very few clicks (i.e., θ is low), then the incumbent with the higher CTR is better off masking the entrant's CTR, thereby securing the top ad position. The intuition resonates with the insights from the main model. Had the search engine learned the entrant's CTR and it turned out to be high, the strong incumbent would risk being "downgraded" to the low-CTR slot below. Conversely, if the second ad slot generates almost as many clicks as the first slot (i.e., high θ), the strong incumbent benefits from the search engine learning the entrant's CTR. The reason is that the incumbent can capitalize on a potentially low c_E realization, while its loss from possibly being driven down to the second ad slot against a high c_E is mitigated by the high θ.

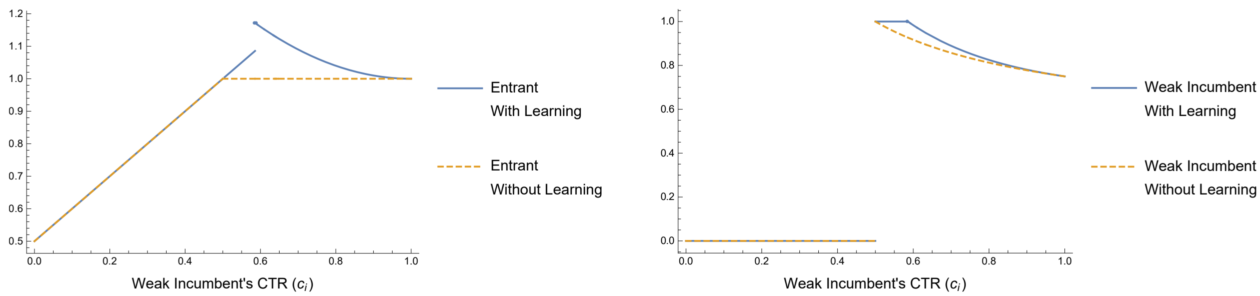
Next, we examine the advertisers' bidding strategies in Stage 1. Figures 7a and 8a depict the entrant's bid with respect to the weak incumbent's CTR, c_i. Observe that for high c_i, the entrant's high bid mirrors the pattern from Section 3 (see Figure 2). Intuitively, if the competing incumbent's CTR is high, then the entrant can earn positive payoffs if and only if (i) its CTR is learned by the search engine in Stage 2 and (ii) the realized CTR turns out to be higher than c_i. Therefore, when facing a strong incumbent, the entrant bids aggressively in Stage 1 in order to create an opportunity to receive a positive payoff in Stage 2.

When facing a low c_i, the entrant's bidding strategy may diverge from the main model. In contrast to the single-slot case, the entrant may lower its bid when the incumbent's CTR is low (see Figure 7a). The intuition revolves around the weak incumbent's incentive to help the search engine learn the entrant's



(a) Entrant *(b) Weak Incumbent*

Figure 7: Optimal bids in GSP without a reserve price; $\delta = 1$, $\theta = \frac{1}{2}$



(a) Entrant *(b) Weak Incumbent*

Figure 8: Optimal bids in GSP with a reserve price; $\delta = 1$, $\theta = \frac{1}{2}$

CTR. Figure 7b shows that a weak incumbent shades its bid in order to help the entrant secure the second slot in Stage 1. The search engine can then learn the entrant's CTR, which in turn creates an opportunity for the weak incumbent to win in Stage 2. And since in the LREF equilibrium the advertisers' bids change in proportion to their competitors', the entrant also shades its bid for low c_i.

However, when the search engine sets a sufficiently high reserve price, the weak incumbent bids below the reserve price in the LREF equilibrium (see Figure 8b). As a result, the entrant's incentive to shade its own bid disappears, and we recover the overbidding pattern from the main model (see Figure 8a). We formalize this finding in the next proposition.

Proposition 11. *The entrant always bids (weakly) higher in the setting where the search engine does not know (but can learn) the entrant's CTR compared to the full-information setting if and only if the reserve price is sufficiently high.*

6 Conclusion

In this paper, we study learning in online advertising. We investigate how a publisher's lack of information about a new advertiser's click-through rate affects the strategies of new and existing advertisers, as well as the publisher. Our theoretical analysis offers useful insights for several issues of managerial importance.

Implications for New Advertisers. We show that when a new advertiser starts online advertising with a publisher, it should bid aggressively in the beginning, sometimes even above its valuation. The reason is that the new advertiser earns a higher expected future payoff when its CTR is learned by the publisher than when it is not. The fact that the new advertiser's CTR can only be learned when the advertiser wins sufficiently many auctions provides strong incentives for the new advertiser to bid aggressively until its CTR is learned.

Our results also indicate that a new advertiser should be prepared to, temporarily, pay more than its valuation per click in the beginning. If the advertiser's CTR turns out to be high, the average cost-per-click will decline over time. In other words, a new advertiser should not leave the market even if the initial cost of advertising is high.

Implications for Existing Advertisers. The entry of a new advertiser has two negative effects for an existing advertiser. First, if the new advertiser's CTR turns out to be high, the existing advertiser risks losing its ad slot to the new advertiser. Second, since the online advertising slots are sold in auctions, entry of a new advertiser increases the payment of the existing advertiser. We demonstrate that, in response to these entry effects, an existing advertiser with a high CTR — e.g., a trademark owner advertising on its branded keywords, or a manufacturer advertising on its own product pages on an online retailer — should bid more aggressively to make it harder for the new advertiser to reveal its CTR. On the other hand, an existing advertiser with a low CTR — e.g., lowest-slot advertisers —should lower its bid to make the revelation process easier. By doing so, the existing advertiser foregoes its short-term profit, but creates an opportunity to earn a larger long-term profit in the event that the new advertiser's CTR turns out to be low.

Implications for the Publisher. When a new advertiser enters the market, the publisher does not know its CTR; the CTR can only be learned if the new advertiser's ad is displayed to consumers sufficiently many times. On the surface, it appears that this lack of information about the new advertiser would lead to a suboptimal allocation of the ad slot, and thus lower the publisher's expected revenue. Surprisingly, our result shows that the ignorance may be a boon to the publisher: its ignorance may incentivize the advertisers to bid more aggressively, which in turn may increase the publisher's revenue compared to the full information benchmark.

The publisher's ignorance, however, is not always blissful. In particular, if the existing advertiser's CTR is high, the lack of information about the new advertiser may hurt the publisher's long-term revenue. We show the publisher can mitigate this loss by favoring the new advertiser in the auction. For example, by lowering the reserve price of, offering free ad credit to, or artificially inflating the bid of the new advertiser, the publisher can increase the probability that the new advertiser wins. This allows the publisher to learn the new advertiser's CTR more quickly, which in turn increases the publisher's long-term revenue. In fact, our results show that the optimal selling mechanism favors the new advertiser in the early rounds of the auction.

Future Research. Our work is a first step towards understanding how agents strategically respond to a publisher's learning process. Future research could explore other scenarios where agents and publishers interact in a learning environment. For instance, a publisher may want to learn sellers' qualities of products for ranking purposes, or customers' WTP for pricing purposes. In addition, while we allow the transition from an incomplete to a full information game, we make several simplifying assumptions in doing so. For example, the transition is discrete and binary in our model. Analyzing the advertisers' strategies in a model with gradual, continuous learning process could lead to interesting additional insights. Finally, our model assumes that advertisers' CTRs are exogenously given. While this assumption is realistic for a given ad copy, it does not capture advertisers' constant efforts in improving their ad copies (e.g., through experimenting with new ad copies). Modeling advertisers' experiments in improving their CTRs, while the CTRs are being learned by the publisher and the advertisers themselves, is another interesting avenue for future research.

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A Proofs

A.1 Proof of Proposition 1

Proof. Consider the incumbent's second stage payoff as a function of its second stage bid b_{I2}, given the entrant's second stage bid b_{E2}.

$$\pi_{I2}(b_{I2}|b_{E2}) = \begin{cases} c_I \left(1 - \frac{\max[c_E b_{E2}, R]}{c_I}\right) & \text{if } c_I b_{I2} \geq \max[c_E b_{E2}, R], \\ 0 & \text{if } c_I b_{I2} < \max[c_E b_{E2}, R]. \end{cases}$$

We will show that truthful bidding weakly dominates bidding below and above valuation. To that end, suppose the incumbent bids *below* valuation such that $c_I b_{I2} < c_I$. If $\max[c_E b_{E2}, R] \leq c_I$, then truthful bidding ensures a positive payoff of $c_I - \max[c_E b_{E2}, R]$ whereas bidding below valuation yields either the same payoff (if $\max[c_E b_{E2}, R] \leq c_I b_{I2} < c_I$), or a lower payoff of zero (if $c_I b_{I2} < \max[c_E b_{E2}, R] < c_I$). And both strategies yield zero payoff if $c_I < \max[c_E b_{E2}, R]$. Therefore, truthful bidding weakly dominates underbidding.

Next, suppose the incumbent bids *above* valuation such that $c_I b_{I2} > c_I$. If $\max[c_E b_{E2}, R] \leq c_I$, then both strategies yield the same positive payoff of $c_I - \max[c_E b_{E2}, R]$, and if $c_I b_{I2} < \max[c_E b_{E2}, R]$, then both strategies yield zero payoff as the incumbent loses the auction. On the other hand, if $c_I < \max[c_E b_{E2}, R] \leq c_I b_{I2}$, then truthful bidding yields zero payoff whereas overbidding yields a negative payoff of $c_I - \max[c_E b_{E2}, R]$. Therefore, truthful bidding weakly dominates overbidding.

Given the second stage auction outcome, we analyze the first stage auction. The incumbent's first stage payoff as a function of its first stage bid, given the entrant's first and second stage bids b_{E1} and b_{E2} is

$$\pi_{I1}(b_{I1}|b_{E1}) = \begin{cases} c_I \left(1 - \frac{\max[c_E b_{E1}, R]}{c_I}\right) + \delta\pi_{I2}(b_{I2} = 1|b_{E2}) & \text{if } c_I b_{I1} \geq \max[c_E b_{E1}, R], \\ \delta\pi_{I2}(b_{I2} = 1|b_{E2}) & \text{if } c_I b_{I1} < \max[c_E b_{E1}, R], \end{cases}$$

where the second stage payoff $\pi_{I2}(b_{I2} = 1|b_{E2} = 1)$ reflects the advertisers' rational anticipation of truthful bidding equilibrium in the second stage auction.

Since the incumbent's second stage payoff is the same regardless of the outcome of the first stage auction, it is immaterial when the incumbent determines its Stage 1 bid. Therefore, by the same reasoning as above, we can show that a weakly dominant strategy in Stage 1 is also truthful bidding. The weak dominance of truthful bidding strategy for the entrant can be shown in a similar manner and is omitted.

Finally, consider the publisher's revenue. In any stage, the publisher receives $c_I\,(\max[c_E, R]/c_I)$ if the incumbent wins, and $c_E\,(c_I/c_E)$ if the entrant wins. The publisher receives nothing if both advertisers' effective bids are below the reserve price. The result follows. ■

A.2 Proof of Lemma 1

Proof. Since the weak dominance argument of truthful bidding in Stage 2 under full information in Proposition 1 does not rest on any assumption regarding an advertiser's knowledge of its competitor's CTR, the truthful bidding result is preserved under incomplete information. ■

A.3 Proof of Lemma 2

Proof. Consider the incumbent's payoff. If the incumbent wins the first stage auction, then it pays $\max[\mu_E b_{E1}, R]/c_I$ and the entrant's CTR remains unknown. Thus, in the second stage auction, where both advertisers bid truthfully, the incumbent's payoff is $c_I\,(1 - \mu_E/c_I)^+$. On the other hand, if the incumbent does not win in Stage 1, then its Stage 1 payoff is zero; however, it has a chance of winning in Stage 2 in the event that c_E turns out to be very low. Specifically, its second stage payoff conditional on losing the first stage auction is $\int_0^1 c_I\,(1 - \max[c_E, R]/c_I)^+\; dF_E$. In sum, the incumbent's payoff as a function of its bid is

$$\mathbb{E}[\pi_I] = \begin{cases} c_I\left(1 - \frac{\max[\mu_E b_{E1}, R]}{c_I}\right) + \delta c_I \left(1 - \frac{\mu_E}{c_I}\right)^+ & \text{if } c_I b_{I1} \geq \max[\mu_E b_{E1}, R], \\ \delta \int_0^1 c_I \left(1 - \frac{\max[c_E, R]}{c_I}\right)^+ dF_E & \text{if } c_I b_{I1} < \max[\mu_E b_{E1}, R] \text{ and } \mu_E b_{E1} \geq R, \\ \delta c_I \left(1 - \frac{\mu_E}{c_I}\right)^+ & \text{if } c_I b_{I1} < \max[\mu_E b_{E1}, R] \text{ and } \mu_E b_{E1} < R. \end{cases}$$

Suppose $\mu_E b_E < R$. Any effective bid above R yields payoff $c_I - R + \delta\,(c_I - \mu_E)^+$, which is greater than the payoff of any other effective bid less than R: $\delta\,(c_I - \mu_E)^+$. Therefore, any effective bid greater than R weakly dominates for $\mu_E b_E < R$; in particular, if $c_I + \delta\left((c_I - \mu_E)^+ - \int_0^1 (c_I - \max[c_E, R])^+\, dF_E\right) > R$, then a weakly dominant is b_I such that $c_I b_I = c_I + \delta\left((c_I - \mu_E)^+ - \int_0^1 (c_I - \max[c_E, R])^+\, dF_E\right)$. On the other hand, if $\mu_E b_E \geq R$, then a weakly dominant bid must satisfy $c_I + \delta(c_I - \mu_E)^+ - c_I b_I = \delta\,(c_I - \mu_E)^+$. Therefore, for any $\mu_E b_E$, a weakly dominant first stage bid is $b_{I1}^* = 1 + \frac{\delta}{c_I}\left((c_I - \mu_E)^+ - \int_0^{c_I} c_I - \max[c_E, R]\, dF_E\right)$.

Next, consider the entrant's payoff. If the entrant wins in Stage 1, then it pays $\max[c_I b_{I1}, R]/\mu_E$ and the entrant's CTR becomes learned. Therefore, in Stage 2, the entrant's expected payoff is $\int_0^1 c_E\,(1 - c_I/c_E)^+\, dF_E$. On the other hand, if the entrant does not win in Stage 1, then its Stage 1 payoff is zero and its CTR remains unknown. However, it still has a chance of winning in Stage 2 when the incumbent's CTR is sufficiently low. Specifically, the entrant's second stage payoff conditional on losing the first stage auction is $c_E\,(1 - c_I/\mu_E)^+$. In sum, the entrant's payoff as a function of its bid is

$$\mathbb{E}[\pi_E] = \begin{cases} \mu_E\left(1 - \frac{\max[c_I b_{I1}, R]}{\mu_E}\right) + \delta \int_{c_I}^1 c_E \left(1 - \frac{c_I}{c_E}\right) dF_E & \text{if } \mu_E b_{E1} > \max[c_I b_{I1}, R], \\ \delta \mu_E \left(1 - \frac{c_I}{\mu_E}\right)^+ & \text{if } \mu_E b_{E1} \leq \max[c_I b_{I1}, R]. \end{cases} \tag{A.1}$$

Therefore, a weakly dominant first stage bid for the entrant is $b_{E1}^* = 1 + \frac{\delta}{\mu_E}\left(\int_{c_I}^1 c_E - c_I\, dF_E - (\mu_E - c_I)^+\right)$.

■

A.4 Proof of Proposition 2

We first prove two intermediary results which will be used for the proof.

Claim 1. *Suppose a differentiable function $f(x)$ is single-peaked on the interval $[a, b]$ (i.e., there exists some $\xi \in (a, b)$ such that $f'(x) \geq 0$ for all $x \leq \xi$ and $f'(x) \leq 0$ for all $x \geq \xi$) and $f(a) < 0 < f(b)$.*

Then there exists a pair $\tilde{x}_1 \leq \tilde{x}_2$ in (a,b) such that (i) $f(x) < 0$ for all $x \in [a, \tilde{x}_1)$, (ii) $f(x) = 0$ for all $x \in [\tilde{x}_1, \tilde{x}_2]$, and (iii) $f(x) > 0$ for all $x \in (\tilde{x}_2, b]$.

Proof of Claim 1. By the Intermediate Value Theorem (IVT), there must exist at least one root in the interval (a,b). From the set of roots (which could be a singleton), let $\tilde{x}_1$ be the smallest; i.e., $\tilde{x}_1 \equiv \min\{x \in (a,b) : f(x) = 0\}$. By definition, we have $f(x) < 0$ for all $x \in [a, \tilde{x}_1)$. Next, let $\tilde{x}_2 \equiv \min\{x \in [\tilde{x}_1, b) : f'(x) > 0\}$. The existence of $\tilde{x}_2$ is guaranteed by $f(b) > 0$ and continuity of $f(x)$. Then $f(x) = 0$ for all $x \in [\tilde{x}_1, \tilde{x}_2]$. To see this, suppose to the contrary that either $f(x) < 0$ or $f(x) > 0$ for some $x' \in [\tilde{x}_1, \tilde{x}_2]$. In the first case, single-peakedness implies $f(x) < 0$ for all $x \geq x'$, which contradicts $f(b) > 0$. In the second case, continuity of $f(x)$ implies that there exists some $x'' \in [\tilde{x}_1, x')$ such that $f'(x) > 0$. This contradicts the definition of $\tilde{x}_2$.

Finally, $f(x) > 0$ for all $x \in (\tilde{x}_2, b]$. For otherwise, if $f(x) \leq 0$ for some $x \in (\tilde{x}_2, b]$, then it must be that $f(x)$ had crossed the x-axis from above for some $x''' \in (\tilde{x}_2, b]$; but then, single-peakedness implies $f(x) < 0$ for all $x > x'''$, which again contradicts $f(b) > 0$. This completes the proof. ■

Claim 2. *If F_E is continuous, then $\frac{\partial}{\partial c_I} \int_{c_I}^1 1 - \frac{c_E}{c_I} dF_E = \frac{1}{c_I^2} \int_{c_I}^1 c_E \, dF_E$.*

Proof of Claim 2. At $c_I = 1$, we have $\frac{\partial}{\partial c_I} \int_{c_I}^1 1 - \frac{c_E}{c_I} \, dF_E = \frac{\partial}{\partial c_I} 0 = 0 = \frac{1}{(1)^2} \int_1^1 c_E \, dF_E$. Next, suppose $c_I \in [0,1)$. Then $\frac{\partial}{\partial c_I} \int_{c_I}^1 1 - \frac{c_E}{c_I} \, dF_E$ is equal to

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\int_{c_I + \varepsilon}^1 1 - \frac{c_E}{c_I + \varepsilon} \, dF_E - \int_{c_I}^1 1 - \frac{c_E}{c_I} \, dF_E \right) = \lim_{\varepsilon \to 0} \left(\int_{c_I + \varepsilon}^1 \frac{c_E}{c_I(c_I + \varepsilon)} \, dF_E - \frac{1}{\varepsilon} \int_{c_I}^{c_I + \varepsilon} 1 - \frac{c_E}{c_I} \, dF_E \right). \tag{A.2}$$

The limit operator can be distributed inside the bracket because the limit values of the summands converge. For the first summand, $\lim_{\varepsilon \to 0} \int_{c_I+\varepsilon}^1 \frac{c_E}{c_I(c_I+\varepsilon)} \, dF_E = \frac{1}{c_I^2} \int_{c_I}^1 c_E \, dF_E$. On the other hand, the second summand converges to zero. To see this, note that the right limit (i.e., $\varepsilon \to 0^+$) of the second summand is "squeezed" from below and above by values which converge to zero:

$$\frac{1}{\varepsilon} \int_{c_I}^{c_I+\varepsilon} 1 - \frac{c_I + \varepsilon}{c_I} \, dF_E \leq \frac{1}{\varepsilon} \int_{c_I}^{c_I+\varepsilon} 1 - \frac{c_E}{c_I} \, dF_E \leq \frac{1}{\delta} \int_{c_I}^{c_I+\varepsilon} 1 - \frac{c_I}{c_I} \, dF_E. \tag{A.3}$$

The left-most term of (A.3) simplifies to $-\int_{c_I}^{c_I+\varepsilon} \frac{1}{c_I} \, dF_E = -\frac{1}{c_I}(F(c_I + \varepsilon) - F(c_I))$, which vanishes to zero as $\varepsilon \to 0^+$ due to right-continuity of F_E. The right-most term also vanishes to zero because the term inside the integral is zero. Similarly, the left limit (i.e., $\varepsilon \to 0^-$) can be shown to converge to zero by bounding the integral from below and above and then invoking the continuity assumption. Therefore, $(A.2) = \lim_{\varepsilon\to 0} \int_{c_I+\varepsilon}^1 \frac{c_E}{c_I(c_I+\varepsilon)}\, dF_E - \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{c_I}^{c_I+\varepsilon} 1 - \frac{c_E}{c_I}\, dF_E = \frac{1}{c_I^2} \int_{c_I}^1 c_E\, dF_E$. ■

Proof of Proposition 2. Consider the incumbent's bid. Whether the bid is below or above valuation depends on the sign of $g(c_I) \equiv \left(1 - \frac{\mu_E}{c_I}\right)^+ - F_E(R)\left(1 - \frac{R}{c_I}\right)^+ - \int_R^1 \left(1 - \frac{c_E}{c_I}\right)^+ dF_E$. If $0 \le c_I \le R$, then $g(c_I) = 0$, so the incumbent bids truthfully. If $R < c_I \le \mu_E$, then $g(c_I) = -F_E(R)\left(1 - \frac{R}{c_I}\right) - \int_R^{c_I} 1 - \frac{c_E}{c_I}\, dF_E$. And since $0 < \mathbb{P}\{c_E \le R\}$, we have $g(c_I) < 0$, which means that the incumbent bids below valuation. Finally, if $\mu_E < c_I \le 1$, then

$$g(c_I) = \left(1 - \frac{\mu_E}{c_I}\right) - F_E(R)\left(1 - \frac{R}{c_I}\right) - \int_R^{c_I} 1 - \frac{c_E}{c_I} dF_E. \tag{A.4}$$

We will show that (A.4) satisfies the properties of Claim 1, thereby proving that there exists a pair of thresholds $\tilde{c}_1 \le \tilde{c}_2$ in $(\mu_E, 1)$ that satisfies the properties stated in the proposition.

(i) Differentiability

$$\begin{aligned} g'(c_I) &= \frac{\partial}{\partial c_I}\left(1 - \frac{\mu_E}{c_I} - \int_0^R 1 - \frac{R}{c_I}\, dF_E - \int_R^{c_I} 1 - \frac{c_E}{c_I}\, dF_E\right) \\ &= -\frac{1}{c_I^2}\int_0^1 \max[R - c_E, 0]\, dF_E + \frac{\partial}{\partial c_I}\int_{c_I}^1 1 - \frac{c_E}{c_I}\, dF_E \qquad (\because R < c_I) \\ &= -\frac{1}{c_I^2}\int_0^1 \max[R - c_E, 0]\, dF_E + \frac{1}{c_I^2}\int_{c_I}^1 c_E\, dF_E, \end{aligned} \tag{A.5}$$

where the last equality follows from Claim 2. Since the derivative is well-defined for all $c_I \in (0,1)$, we conclude that $g(c_I)$ is differentiable.

(ii) Single-peakedness

From (A.5), it follows that the sign of $g'(c_I)$ is equal to the sign of $h(c_I) \equiv \int_{c_I}^1 c_E\, dF_E - \int_0^1 \max[R - c_E, 0]\, dF_E$. At $c_I = 0^+$[33], h is positive because $\int_{c_I}^1 c_E\, dF_E - \int_0^1 \max[R - c_E, 0]\, dF_E = \int_{0^+}^1 c_E\, dF_E - \int_0^1 \max[R - c_E, 0]\, dF_E = \int_0^1 c_E - \max[R - c_E, 0]\, dF_E \ge \int_0^1 c_E - R\, dF_E = \mu_E - R > 0$. At $c_I = 1$, h is negative because $\int_{c_I}^1 c_E\, dF_E - \int_0^1 \max[R - c_E, 0]\, dF_E = \int_1^1 c_E\, dF_E - \int_0^1 \max[R - c_E, 0]\, dF_E = -\int_0^1 \max[R - c_E, 0]\, dF_E < 0$. Finally, $h(c_I)$ is non-increasing because for any $\delta > 0$, $h(c_I + \delta) - h(c_I) = \int_{c_I+\delta}^1 c_E\, dF_E - \int_{c_I}^1 c_E\, dF_E = \int_{c_I+\delta}^1 c_E\, dF_E - \int_{c_I+\delta}^1 c_E\, dF_E - \int_{c_I}^{c_I+\delta} c_E\, dF_E =$

[33] We evaluate at the right-limit 0^+ because $g'(c_I)$ is undefined at $c_I = 0$.

$-\int_{c_I}^{c_I+\delta} c_E \, dF_E \leq 0$.

In total, since $h(c_I)$ is non-increasing in $[0,1]$, $h(0^+) \geq 0$, and $h(1) \leq 0$, by the IVT, there exists a $\xi \in (0,1)$ such that $h(c_I) \geq 0$ for all $c_I \leq \xi$ and $h(c_I) \leq 0$ for all $c_I \geq \xi$. By the sign equivalence, we have $g'(c_I) \geq 0$ for all $c_I \leq \xi$ and $g'(c_I) \leq 0$ for all $c_I \geq \xi$.

(iii) Endpoint values

We have $g(\mu_E) = -F_E(R)\left(1 - \frac{R}{\mu_E}\right) - \int_R^{\mu_E}\left(1 - \frac{c_E}{\mu_E}\right) dF_E < 0$ and $g(1) = 1 - \mu_E - F_E(R)(1-R) - \int_R^1 1 - c_E \, dF_E = 1 - \mu_E - \int_0^R 1 - R \, dF_E - \int_R^1 1 - c_E \, dF_E = \int_0^R R - c_E \, dF_E > 0$.

Therefore, $g(c_I)$ satisfies the properties of Claim 1, which implies that there exists a pair $\tilde{c}_1 \leq \tilde{c}_2$ in $(\mu_E, 1)$ such that $g(c_I) < 0$ for all $c_I \in (\mu_E, \tilde{c}_1)$, $g(c_I) = 0$ for all $c_I \in [\tilde{c}_1, \tilde{c}_2]$, and $g(c_I) > 0$ for all $c_I \in (\tilde{c}_2, 1]$. This, in turn, implies that the incumbent bids below valuation, truthfully, and above valuation for $c_I \in (\mu_E, \tilde{c}_1]$, $c_I \in [\tilde{c}_1, \tilde{c}_2]$, and $c_I \in (\tilde{c}_2, 1]$, respectively.

Second, consider the entrant's bid $b^*_{E1}(c_I) = 1 + \frac{\delta}{\mu_E}\left(\int_{c_I}^1 (c_E - c_I) \, dF_E - (\mu_E - c_I)^+\right)$. Whether the entrant bids below or above valuation depends on the sign of $k(c_I) \equiv \int_{c_I}^1 (c_E - c_I) \, dF_E - (\mu_E - c_I)^+$. If $\mu_E \leq c_I$, then $k(c_I) = \int_{c_I}^1 c_E - c_I \, dF_E \geq 0$, where the inequality holds with equality only if $c_I = 1$. If $\mu_E > c_I$, then $k(c_I) = \int_{c_I}^1 (c_E - c_I) \, dF_E - (\mu_E - c_I)^+ = \int_0^1 c_E - c_I \, dF_E - \int_0^{c_I} c_E - c_I \, dF_E - (\mu_E - c_I) = -\int_0^{c_I} c_E - c_I \, dF_E > 0$. This completes the proof. ■

A.5 Proof of Proposition 3

Proof. First, we will show that there exists a unique $\hat{c} \in (\mu_E, 1)$ such that the entrant wins the first stage auction for $c_I < \hat{c}$, and the incumbent wins otherwise. To that end, consider the difference in effective bids $D_A(c_I) \equiv c_I b^*_{I1}(c_I) - \mu_E b^*_{E1}(c_I)$. If $c_I \leq R$, then $D_A(c_I) = c_I - \mu_E + \delta\left(\mu_E - R - \int_R^1 c_E - R \, dF_E\right) = c_I - \mu_E + \delta\left(\int_0^1 c_E - R \, dF_E - \int_R^1 c_E - R \, dF_E\right) = -(\mu_E - c_I) - \delta \int_0^R R - c_E \, dF_E < 0$. Therefore, the entrant wins the first stage auction for all $c_I \leq R$. Note that the entrant also beats the reserve price because $b^*_{E1}(c_I) \geq 1$ (cf. Lemma 2) and $\mu_E > R$.

If $R < c_I < \mu_E$, then $D_A(c_I) = c_I - \mu_E - \delta \int_{c_I}^1 c_E - c_I \, dF_E + \delta\left(c_I - \mu_E - \int_0^1 (c_I - \max[c_E, R])^+ \, dF_E\right) < 0$. Therefore, the entrant wins the first stage auction in this interval as well.

Finally, if $\mu_E \leq c_I \leq 1$, then $D_A(c_I) = (1+\delta)(c_I - \mu_E) - \delta \int_{c_I}^1 c_E - c_I \, dF_E - \delta \int_0^1 (c_I - \max[c_E, R])^+ \, dF_E$.

Thus, $D'_A(c_I) = 1 + \delta(1 - F_E(c_I)) + \delta\left(1 - \frac{\partial}{\partial c_I}\int_0^1 (c_I - \max[c_E, R])^+ \, dF_E\right)$, which simplifies to

$$\begin{cases} 1 + \delta(1 - F_E(c_I)) + \delta\,(1 - 1) & \text{if } c_I \geq \max[c_E, R], \\ 1 + \delta(1 - F_E(c_I)) + \delta\,(1 - 0) & \text{if } c_I < \max[c_E, R], \end{cases} > 0.$$

Therefore, $D_A(c_I)$ is strictly increasing in the interval $[\mu_E, 1]$. Combined with the fact that $D_A(\mu_E) = -\delta\int_{\mu_E}^1 c_E - \mu_E \, dF_E - \delta\int (\mu_E - \max[c_E, R])^+ \, dF_E < 0$ and $D_A(1) = 1 - \mu_E + \delta\left(1 - \mu_E - \int 1 - \max[c_E, R] \, dF_E\right) = 1 - \mu_E + \delta\int \max[c_E, R] - c_E \, dF_E > 0$, we have, by the IVT, a unique $\hat{c} \in (\mu_E, 1)$ such that $D_A(c_I) < 0$ for all $c_I < \hat{c}$ and $D_A(c_I) > 0$ for all $c_I > \hat{c}$. More generally, combining the results from the intervals above yield that the entrant wins the first stage bid for all $c_I < \hat{c}$ and the incumbent wins for all $c_I \geq \hat{c}$.

Next, we characterize the publisher's expected payoff when it knows the entrant's CTR before the auctions take place. If the entrant's CTR is known, then advertisers bid truthfully in each round. Therefore, the publisher's total expected revenue under full information is $\mathbb{E}[\pi_P^F] = (1+\delta)\left(\int_{c_I}^1 c_I \, dF_E + \int_0^{c_I} \max[c_E, R] \, dF_E\right)$. On the other hand, if c_E is *a priori* unknown to all players and is revealed only if the entrant wins an auction, then the publisher's expected revenue is

$$\mathbb{E}[\pi_P] = \begin{cases} \mu_E b_{E1}^* + \delta \max[\min[c_I, \mu_E], R] & \text{if } c_I b_{I1}^* \geq \mu_E b_{E1}^*, \\ \max[c_I b_{I1}^*, R] + \delta\left(\int_m^1 m \, dF_E + \int_0^m \max[c_E, R]\mathbb{I}_{\{c_I \geq \max[c_E, R]\}} \, dF_E\right) & \text{if } c_I b_{I1}^* < \mu_E b_{E1}^*. \end{cases} \quad \text{(A.6)}$$

Next, define the difference $D_\pi(c_I) \equiv \mathbb{E}[\pi_P] - \mathbb{E}[\pi_P^F]$. The set of c_I for which $D_\pi(c_I) > 0$ corresponds to the region where the publisher's revenue is higher when it does not know c_E than when it does.

If $R < c_I \leq \mu_E$, then $D_\pi(c_I) = \max\left[c_I - \delta\int_0^{c_I} c_I - \max[c_E, R] \, dF_E, R\right] - \left(\int_0^{c_I} \max[c_E, R] \, dF_E + \int_{c_I}^1 c_I \, dF_E\right)$, which is positive if and only if

$$c_I - \int_R^{c_I} c_E \, dF_E - \int_{c_I}^1 c_I \, dF_E - \int_0^R R \, dF_E - \delta\left(\int_R^{c_I} c_I - c_E \, dF_E + \int_0^R c_I - R \, dF_E\right) > 0.$$

But the expression on the left-hand side is 0 at $c_I = R$, and its derivative with respect to c_I is $(1 - \delta)F_E(c_I)$. This implies that if $\delta < 1$, then $D_\pi(c_I) > 0$ for all $R < c_I \leq \mu_E$, and if $\delta > 1$, then $D_\pi(c_I) < 0$ for all $R < c_I \leq \mu_E$.

If $\mu_E < c_I < \hat{c}$, then

$$D_\pi(c_I) = \max\left[c_I + \delta\left(c_I - \mu_E - \int_0^{c_I} c_I - \max[c_E, R]\, dF_E\right), R\right] - \left(\int_0^{c_I} \max[c_E, R]\, dF_E + \int_{c_I}^1 c_I\, dF_E\right),$$

which is positive if and only if

$$c_I - \int_R^{c_I} c_E\, dF_E - \int_{c_I}^1 c_I\, dF_E - \int_0^R R\, dF_E + \delta\left(c_I - \mu_E - \int_R^{c_I} c_I - c_E\, dF_E - \int_0^R c_I - R\, dF_E\right) > 0.$$

Now, the expression on the left-hand side is strictly increasing in c_I because $\frac{\partial}{\partial c_I} LHS = F_E(c_I) + \delta(1 - F_E(c_I)) > 0$. Furthermore, the difference is negative and positive at $c_I = R$ and $c_I = 1$, respectively: $D_\pi(R) = \delta(R - \mu_E) < 0$ and $D_\pi(1) = 1 - \int_R^1 c_E\, dF_E - \int_0^R R\, dF_E + \delta\left(1 - \mu_E - \int_R^1 1 - c_E\, dF_E - \int_0^R 1 - R\, dF_E\right) = 1 - \int_0^1 \max[c_E, R]\, dF_E + \delta \int_0^1 \max[c_E, R] - c_E\, dF_E > 0$. Therefore, by the IVT, there exists a unique $\tilde{c}_1 \in (r, 1)$ such that $D_\pi(c_I) < 0$ for all $c_I \in (R, \tilde{c}_1)$ and $D_\pi(c_I) > 0$ for all $c_I \in (\tilde{c}_1, 1)$. However, the interval in question here is $(\mu_E, \hat{c})$, so we re-define the threshold as $\underline{c} \equiv \max[\mu_E, \min[\hat{c}, \tilde{c}_1]]$.

Finally, if $\hat{c} \le c_I \le 1$, then the incumbent wins the first stage auction and the difference in payoffs between the uncertain and full information cases is

$$D_\pi(c_I) = (1 + \delta)\left(\mu_E - \left(\int_0^R R\, dF_E + \int_R^{c_I} c_E\, dF_E + \int_{c_I}^1 c_I\, dF_E\right)\right) + \delta \int_{c_I}^1 c_E - c_I\, dF_E.$$

Similarly as above, we invoke the IVT to prove the unique existence of a root. We have $D'_\pi(c_I) = -(1 + 2\delta)(1 - F_E(c_I)) < 0$,

$$\begin{aligned} D_\pi(\mu_E) &= (1 + \delta)\left(\mu_E - \left(\int_0^R R\, dF_E + \int_R^{\mu_E} c_E\, dF_E + \int_{\mu_E}^1 \mu_E\, dF_E\right)\right) + \delta \int_{\mu_E}^1 c_E - \mu_E\, dF_E \\ &= (1 + \delta)\left(\int_R^{\mu_E} \mu_E - c_E\, dF_E + \int_0^R \mu_E - R\, dF_E\right) + \delta \int_{\mu_E}^1 c_E - \mu_E\, dF_E > 0, \end{aligned}$$

and $D_\pi(1) = (1 + \delta)\left(\mu_E - \int_R^1 c_E\, dF_E - \int_0^R R\, dF_E\right) = (1 + \delta)\int_0^R c_E - R\, dF_E < 0$. Therefore, by the IVT, there exists a unique $\tilde{c}_2 \in (\mu_E, 1)$ such that $D_\pi(c_I) > 0$ for all $c_I \in (\mu_E, \tilde{c}_2)$ and $D_\pi(c_I) < 0$ for all $c_I \in (\tilde{c}_2, 1)$. However, the interval in question here is $[\hat{c}, 1]$, so we bound the threshold as $\bar{c} \equiv \max[\hat{c}, \tilde{c}_2]$. Putting together all the sets for which $D_\pi > 0$ yields the result. ∎

A.6 Proof of Lemma 3

Proof. The Stage 2 optimal mechanism, (Q_2^*, M_2^*), can be obtained by directly applying Myerson's Lemma (Myerson, 1981) and is omitted. We solve for the Stage 1 optimal mechanism.

First, we introduce notations. For $j \in \{I, E\}$, let $U_j(x_{j1})$ denote advertiser j's Stage 1 utility (in particular, accounting for expected Stage 2 utility as well) under a generic mechanism (Q_1, M_1), when advertiser j's type draw in Stage 1 is $x_{j1} \in \left[\frac{\delta\Delta_j}{c_j}, \overline{v}_j + \frac{\delta\Delta_j}{c_j}\right]$. That is, under (Q_1, M_1), if advertiser j's type draw is x_{j1} and advertiser k's is x_{k1}, then advertiser j wins the ad slot in Stage 1 with probability $Q_{j1}(x_{j1}, x_{k1})$ and pays (per-click) $M_{j1}(x_{j1}, x_{k1})$. The distribution of advertiser j's type follows $H_j(x_{j1}) = G_j\left(x_{j1} - \frac{\delta\Delta_j}{c_j}\right)$. To facilitate exposition, we slightly abuse notation and write the publisher's Stage 2 estimate of entrant's CTR (which is c_E if the publisher learns it in Stage 2 and μ_E otherwise) as c_E. Let $q_{I1}(z_I) = \int_0^{\overline{v}_E} Q_{I1}(z_I, x_{E1})\, dH_E$ be the probability that incumbent believes *incumbent* will win if it bids z_I, $q_{E1}^I(z_I) = \int_0^{\overline{v}_E} Q_{E1}(z_I, x_{E1})\, dH_E$ the probability that incumbent believes *entrant* will win if it bids z_I, and $q_{E1}^E(z_E) = \int_0^{\overline{v}_I} Q_{E1}(x_{I1}, z_E)\, dH_I$ the probability that entrant believes entrant will win if it bids z_E, where $H_j(x_{j1}) = G_j\left(x_{j1} - \frac{\delta\Delta_j}{c_j}\right)$.

We will derive expressions for the expected payments from the advertisers to the publisher under the generic (Q_1, M_1), which is incentive compatible (IC) and individually rational (IR), and then solve for the publisher's problem of optimizing (Q_1, M_1).

We have

$$\begin{aligned}
U_{I1}(x_{I1}) = {} & q_{I1}(x_{I1})\left(c_I x_{I1} + \delta \int_0^1 \pi_{I2}(c_E)\, dF_E\right) && \text{(incumbent wins)} \\
& + q_{E1}^I(x_{I1})\delta\left(0 + \int_0^1 \pi_{I2}(c_E)\, dF_E\right) && \text{(entrant wins)} \\
& + (1 - q_{I1}(x_{I1}) - q_{E1}^I(x_{I1}))\,(0 + \delta\pi_{I2}(\mu_E)) && \text{(nobody wins)} \\
& - c_I m_{I1}(x_{I1}) \\
= {} & q_{I1}(x_{I1})(c_I x_{I1} - \delta\Delta_I) - q_{E1}^I(x_{I1})\delta\Delta_I + \delta\pi_{I2}(\mu_E) - c_I m_{I1}(x_{I1})
\end{aligned} \tag{A.7}$$

and

$$
\begin{aligned}
U_{E1}(x_E) &= q_{E1}^E(x_{E1})\left(\mu_E x_{E1} + \delta\pi_{E2}(\mu_E)\right) && \text{(entrant wins)} \\
&\quad + (1 - q_{E1}^E(x_{E1}))(0 + \delta\pi_{E2}(\mu_E)) && \text{(otherwise)} \\
&\quad - \mu_E m_{E1}(x_{E1}) \\
&= q_{E1}^E(x_{E1})\mu_E x_{E1} + \delta\pi_{E2}(\mu_E) - \mu_E m_{E1}(x_{E1}). && \text{(A.8)}
\end{aligned}
$$

Next, we re-write these profit expressions using IC and IR conditions.

Claim 3. *If (Q_1, M_1) is IC, then $U'_{j1}(x_{j1}) = c_j q_{j1}(x_{j1})$ for $j \in \{I, E\}$.*[34]

Thus, we can write

$$
U_{j1}(x_{j1}) = U_{j1}\left(\frac{\delta\Delta_j}{c_j}\right) + \int_{\frac{\delta\Delta_j}{c_j}}^{x_{j1}} c_j q_{j1}(t_j)\, dt_j. \tag{A.9}
$$

Combining equations (A.7), (A.8), and (A.9), we obtain

$$
c_I m_{I1}(x_{I1}) = q_{I1}(x_{I1})(c_I x_{I1} - \delta\Delta_I) - q_{E1}^I(x_{I1})\delta\Delta_I + \delta_I\pi_{I2}(\mu_E) - U_{I1}\left(\frac{\delta\Delta_I}{c_I}\right) - \int_{\frac{\delta\Delta_I}{c_I}}^{x_{I1}} c_I q_{I1}(t_I)\, dt_I \tag{A.10}
$$

and

$$
\mu_E m_{E1}(x_{E1}) = q_{E1}^E(x_{E1})\mu_E x_{E1} + \delta\pi_{E2}(\mu_E) - U_{E1}\left(\frac{\delta\Delta_E}{\mu_E}\right) - \int_{\frac{\delta\Delta_E}{\mu_E}}^{x_{E1}} c_E q_{E1}(t_E)\, dt_E. \tag{A.11}
$$

We can simplify $U_{j1}\left(\frac{\delta\Delta_j}{c_j}\right)$ by invoking IR. From (A.7) and (A.8), we have $U_{I1}\left(\frac{\delta\Delta_I}{c_I}\right) = -q_{E1}^I\left(\frac{\delta\Delta_I}{c_I}\right)\delta\Delta_I + \delta\pi_{I2}(\mu_E) - c_I m_{I1}\left(\frac{\delta\Delta_I}{c_I}\right)$ and $U_{E1}\left(\frac{\delta\Delta_E}{\mu_E}\right) = q_{E1}^E\left(\frac{\delta\Delta_E}{\mu_E}\right)\delta\Delta_E + \delta\pi_{E2}(\mu_E) - \mu_E m_{E1}\left(\frac{\delta\Delta_E}{\mu_E}\right)$. Substitution of these values into (A.10) and (A.11) implies that the payment function M_1 enters the publisher's objective function only as $m_{j1}\left(\frac{\delta\Delta_j}{c_j}\right)$ for $j \in \{I, E\}$. And since the objective function increases in $m_{j1}\left(\frac{\delta\Delta_j}{c_j}\right)$, the optimal mechanism M_1 will be such that $m_{j1}(0)$ is set as high as IR permits.

To satisfy IR, we must have $U_{j1}(x_{j1}) \geq U_{j0}$ for all x_{j1}, where U_{j0} represents the value of advertiser j's outside option. But from Claim 3, we have that U_{j1} is increasing; therefore, the IR condition simplifies to $U_{j1}\left(\frac{\delta\Delta_j}{c_j}\right) \geq U_{j0}$ under IC. Thus, under the optimal mechanism, $m_{j1}\left(\frac{\delta\Delta_j}{c_j}\right)$ will be set such that

[34] The proof of Claim 3 can be found at the end of this section.

IR binds: $U_{j1}\left(\frac{\delta\Delta_j}{c_j}\right) = U_{j0}$. More explicitly, the outside options for the incumbent and the entrant, respectively, can be written as

$$U_{I0} = q_{E1}^{-I}\delta \int \pi_{I2}(c_E)\, dF_E + \left(1 - q_{E1}^{-I}\right) \delta\pi_{I2}(\mu_E) = \delta\pi_{I2}(\mu_E) - q_{E1}^{-I}\delta\Delta_I \tag{A.12}$$

$$U_{E0} = \delta\pi_{E2}(\mu_E) \tag{A.13}$$

where $q_{E1}^{-I} = \int_0^{\overline{v}_E} Q_{E1}^{-I}(x_{E1})\, dH_E$ is the probability the publisher allocates the ad slot to the entrant when the entrant is the *only* bidder.

Substituting these outside option values, (A.12) and (A.13), into (A.10) and (A.11) yields

$$c_I m_{I1}(x_{I1}) = q_{I1}(x_{I1})(c_I x_{I1} - \delta_I \Delta_I) + \left(q_{E1}^{-I} - q_{E1}^{I}(x_{I1})\right) \delta_I \Delta_I - \int_{\frac{\delta_I \Delta_I}{c_I}}^{x_{I1}} c_I q_{I1}(t_I)\, dt_I, \tag{A.14}$$

$$\mu_E m_{E1}(x_{E1}) = q_{E1}^{E}(x_{E1})\mu_E x_{E1} - \int_0^{x_{E1}} \mu_E q_{E1}^{E}(t_E)\, dt_E. \tag{A.15}$$

Finally, the publisher's objective function can be written as the sum of (i) the expected payments in Stage 1 from the advertisers, and (ii) expected profits in Stage 2 under (Q_2^*, M_2^*) (which are contingent on Stage 1 outcome):

$$\begin{aligned} V(Q_1, M_1) = \iint \Bigg(& Q_{I1}(x_{I1}, x_{E1}) c_I \left(x_{I1} - \frac{\delta\Delta_I}{c_I} - \frac{1 - H_I(x_{I1})}{h_I(x_{I1})} + \frac{\delta}{c_I}\pi_{P2}(\mu_E) \right) \\ & + Q_{E1}(x_{I1}, x_{E1})\mu_E \left(x_{E1} - \frac{\delta\Delta_I}{\mu_E} - \frac{1 - H_E(x_{E1})}{h_E(x_{E1})} + \frac{\delta}{\mu_E} \int \pi_{P2}(c_E)\, dF_E \right) \\ & + (1 - Q_{I1}(x_{I1}, x_{E1}) - Q_{E1}(x_{I1}, x_{E1}))\, \delta\pi_{P2}(\mu_E) \Bigg) + Q_{E1}^{-I}(x_{E1})\delta\Delta_I\, d\mathbf{G}, \end{aligned}$$

This simplifies to

$$\begin{aligned} V(Q_1, M_1) = \iint \Bigg(& Q_{I1}(x_{I1}, x_{E1}) c_I \left(x_{I1} - \frac{\delta\Delta_I}{c_I} - \frac{1 - H_I(x_{I1})}{h_I(x_{I1})} \right) \\ & + Q_{E1}(x_{I1}, x_{E1})\mu_E \left(x_{E1} - \frac{\delta\Delta_I}{\mu_E} - \frac{1 - H_E(x_{E1})}{h_E(x_{E1})} + \frac{\delta}{\mu_E}\Delta_P \right) \Bigg) \\ & + Q_{E1}^{-I}(x_{E1})\delta\Delta_I\, d\mathbf{G} + \delta\pi_{P2}(\mu_E). \end{aligned} \tag{A.16}$$

The optimal Q_1^* corresponds to the allocation rule that chooses the maximal coefficients of Q_{I1} and Q_{E1}, provided they are positive, for any given x_{I1} and x_{E1}. Finally, applying the fact that a mechanism is invariant to positive affine transformations of "virtual valuations" yields the results. ■

Proof of Claim 3 . Consider the incumbent's Stage 1 expected utility when it bids z_{I1} given a type draw of x_{I1}: $U_{I1}(z_{I1}|x_{I1}) = q_{I1}(z_{I1})(c_I x_{I1} - \delta\Delta_I) - q^I_{E1}(z_{I1})\delta\Delta_I + \delta\pi_{I2}(\mu_E) - c_I m_{I1}(z_{I1})$. For any ε, IC implies $\frac{U_{I1}(x_{I1}+\varepsilon|x_{I1}+\varepsilon)-U_{I1}(x_{I1}|x_{I1})}{\varepsilon} \leq \frac{U_{I1}(x_{I1}+\varepsilon|x_{I1}+\varepsilon)-U_{I1}(x_{I1}+\varepsilon|x_{I1})}{\varepsilon} = q_{I1}(x_{I1}+\varepsilon)c_I$ and $\frac{U_{I1}(x_{I1}+\varepsilon|x_{I1}+\varepsilon)-U_{I1}(x_{I1}|x_{I1})}{\varepsilon} \geq \frac{U_{I1}(x_{I1}|x_{I1}+\varepsilon)-U_{I1}(x_{I1}|x_{I1})}{\varepsilon} = q_{I1}(x_{I1})c_I$. As the lower and upper bounds converge to $q_{I1}(x_{I1})c_I$ as $\varepsilon \to 0$, we obtain the desired result. The entrant's result can be derived analogously and is omitted. ■

A.7 Proof of Proposition 4

Proof. The entrant overbids if and only if $\Delta_E > 0$; i.e., the entrant earns a higher Stage 2 profit if the publisher learns its CTR. It suffices to show that the entrant's Stage 2 profit is convex in its true CTR, for then Jensen's inequality would imply the desired result. To that end, consider the entrant's Stage 2 profit when the publisher assigns it its true CTR: $\pi_{E2}(c_E) = \int_0^{\overline{v}_E} U_E(x_E|c_E)\,dG_E = \int_0^{\overline{v}_E} c_E \int_0^{x_E} q^*_{E2}(t_E)g_E(t_E)\,dt_E\,dG_E = \int_0^{\overline{v}_E} c_E q^*_{E2}(t_E)(1-G_E(t_E))\,dG_E(t_E) = \int_{R^*_{E2}}^{\overline{v}_E}\int_0^{\overline{v}_I} c_E \mathbb{I}_{\{c_E > \chi(x_I, x_E|c_I)\}}(1-G_E(x_E))\,dG_I\,dG_E$, where $R^*_{E2} = \inf\left\{b \geq 0 : b - \frac{1-G_E(b)}{g_E(b)} \geq 0\right\}$, $\mathbb{I}_{\{\mathcal{E}\}}$ is the indicator function which is equal to 1 if $\mathcal{E}$ is true, and 0 otherwise, and $\chi(x_I, x_E|c_I) = c_I\left(x_I - \frac{1-G_I(x_I)}{g_I(x_I)}\right)/\left(x_E - \frac{1-G_E(x_E)}{g_E(x_{E1})}\right)$. Since χ is independent of c_E, we obtain that for any given x_I and x_E, $c_E\mathbb{I}_{\{c_E > \xi(x_I, x_E|c_I)\}}$ is convex in c_E. And since any linear combination with positive weights of convex functions is also convex, we conclude that $\pi_{E2}(c_E)$ is convex in c_E. This proves the entrant's result.

Next, we turn to the incumbent. We will work with the following subgradient argument:

Claim 4. *Let $\mathbb{E}[X] = \mu$. If a function $f(x)$ that is differentiable at $x = \mu$ satisfies $f(x) \geq f'(\mu)(x - \mu) + f(\mu) \geq f(x)$ for all x, then $\mathbb{E}[f(X)] \geq f(\mathbb{E}[X])$.*

This follows immediately from $\mathbb{E}[f(X)] \geq \mathbb{E}[f'(\mu)(X-\mu) + f(\mu)] = f'(\mu)\mathbb{E}[X-\mu] + f(\mu) = f(\mu)$.

The incumbent's Stage 2 profit when the publisher assigns c_E is $\pi_{I2}(c_E) = \int_0^{\overline{v}_I} U_I(x_I|c_E, c_I)\,dG_I = \int_0^{\overline{v}_I} c_I \int_0^{x_I} q^*_{I2}(t_I)g_I(t_I)\,dt_I\,dG_I = \int_0^{\overline{v}_I} c_I q^*_{I2}(x_I)(1-G_I(x_I))\,dG_I$, which simplifies to

$$\begin{cases} \int_{R^*_{I2}}^{\overline{v}_I} c_I(1-G_I(x_I))z\left(\frac{c_I\eta_I(x_I)}{c_E}\right)\,dG_I & \text{if } \frac{c_I\overline{v}_I}{\overline{v}_E} < c_E, \\ \int_{R^*_{I2}}^{h_I^{-1}\left(\frac{c_E\overline{v}_E}{c_I}\right)} c_I(1-G_I(x_{I1}))z\left(\frac{c_I\eta_I(x_I)}{c_E}\right)\,dG_I + \int_{h_I^{-1}\left(\frac{c_E\overline{v}_E}{c_I}\right)}^{\overline{v}_I} c_I(1-G_I(x_I))\,dG_I & \text{otherwise,} \end{cases}$$

where $z(y) = G_E\left(\eta_E^{-1}(y)\right)$, and $\eta_j(x) = x - \frac{1-G_j(x)}{g_j(x)}$ for $j \in \{I, E\}$.

Suppose $c_I < \frac{\overline{v}_E \mu_E}{\overline{v}_I}$ such that at $\pi_{I2}(\mu_E) = \int_{R^*_{I2}}^{\overline{v}_I} c_I(1 - G_I(x_I)) z\left(\frac{c_I \eta_I(x_I)}{\mu_E}\right) dG_I$ and $\pi'_{I2}(\mu_E) = \int_{R^*_{I2}}^{\overline{v}_I} c_I(1 - G_I(x_I)) z'\left(\frac{c_I \eta_I(x_I)}{\mu_E}\right)\left(-\frac{c_I \eta_I(x_I)}{\mu_E^2}\right) dG_I$. By Claim 4, it suffices to show that

$$\pi_{I2}(c_E) \geq \pi'_{I2}(\mu_E)(c_E - \mu_E) + \pi_{I2}(\mu_E) \text{ for all } c_E. \tag{A.17}$$

Note that $z(\cdot) \leq 1$, which implies that

$$\int_{h_I^{-1}\left(\frac{c_E \overline{v}_E}{c_I}\right)}^{\overline{v}_I} c_I(1 - G_I(x_I))\, dG_I \geq \int_{h_I^{-1}\left(\frac{c_E \overline{v}_E}{c_I}\right)}^{\overline{v}_I} c_I(1 - G_I(x_I)) z\left(\frac{c_I \eta_I(x_I)}{c_E}\right) dG_I$$

for all c_E. Therefore, a sufficient condition for (A.17) is that for all c_E,

$$\begin{aligned}\int_{R^*_{I2}}^{\overline{v}_I} (1 - G_I(x_I)) z\left(\frac{c_I \eta_I(x_I)}{c_E}\right) dG_I \geq \int_{R^*_{I2}}^{\overline{v}_I} (1 - G_I(x_I)) \Bigg(z'\left(\frac{c_I \eta_I(x_I)}{\mu_E}\right)\left(-\frac{c_I \eta_I(x_I)}{\mu_E^2}\right)(c_E - \mu_E) \\ + z\left(\frac{c_I \eta_I(x_I)}{\mu_E}\right)\Bigg) dG_I.\end{aligned} \tag{A.18}$$

Finally, a sufficient condition for (A.18) is that $\int_{R^*_{I2}}^{\overline{v}_I} (1 - G_I(x_I)) z\left(\frac{c_I \eta_I(x_I)}{c_E}\right) dG_I$ be convex in c_E for all c_E. The convexity condition simplifies to

$$\int_{R^*_{I2}}^{\overline{v}_I} (1 - G_I(x_I)) z''\left(\frac{c_I \eta_I(x_I)}{c_E}\right) dG_I \geq 0 \text{ for all } c_E. \tag{A.19}$$

Note that $z\left(\frac{c_I \eta_I(x_I)}{c_E}\right) = \mathbb{P}\{c_E \eta_E(x_E) \leq c_I \eta_I(x_I)\}$, which is the probability that the entrant's valuation draw is such that the incumbent wins in Stage 2. This probability can be easily verified to be decreasing in c_E. Now, condition (A.19) can be interpreted as this probability being "sufficiently convex" for all x_I. This is equivalent to the condition that the rate of decline of the incumbent's winning probability in c_E be sufficiently low.

Finally, the sufficient condition for overbidding around the neighborhood of $c_I = 1$ follows immediately from Claim 4 and the continuity of $\pi_{I2}(c_E)$ with respect to c_I:

$$\pi_{I2}(c_E) \leq \pi'_{I2}(\mu_E)(c_E - \mu_E) + \pi_{I2}(\mu_E) \tag{A.20}$$

This means that the incumbent's Stage 2 profit when c_E turns out to be high is considerably low; i.e., the risk of revealing the entrant's CTR is high.[35] ■

A.8 Proof of Proposition 5

Proof. Let R_E^F be the entrant's reserve price under full information. R_E^F satisfies

$$R_E^F - \frac{1 - G_E(R_E^F)}{g_E(R_E^F)} = 0. \tag{A.21}$$

When the publisher does not know the entrant's CTR, the optimal reserve price R_E satisfies

$$R_E - \frac{1 - G_E\left(R_E - \frac{\delta\Delta_E}{\mu_E}\right)}{g_E(R_E - \frac{\delta\Delta_E}{\mu_E})} = \frac{\delta}{\mu_E}(\Delta_I - \Delta_P).$$

Now Assumption 1 implies that for all R,

$$R - \frac{1 - G_E\left(R - \frac{\delta\Delta_E}{\mu_E}\right)}{g_E(R - \frac{\delta\Delta_E}{\mu_E})} < R - \frac{1 - G_E(R)}{g_E(R)}.$$

Therefore, the condition $R_E < R_E^F$ is equivalent to

$$\frac{\delta}{\mu_E}(\Delta_I - \Delta_P) < R_E^F - \frac{1 - G_E\left(R_E^F - \frac{\delta\Delta_E}{\mu_E}\right)}{g_E(R_E^F - \frac{\delta\Delta_E}{\mu_E})}. \tag{A.22}$$

Using (A.21), the right-hand side simplifies to

$$\frac{1 - G_E(R_E^F)}{g_E(R_E^F)} - \frac{1 - G_E\left(R_E^F - \frac{\delta\Delta_E}{\mu_E}\right)}{g_E(R_E^F - \frac{\delta\Delta_E}{\mu_E})},$$

which is negative by Assumption 1. Labeling this negative object as $-\rho$ simplifies (A.22) to $\Delta_P > \Delta_I + \frac{\mu_E\rho}{\delta}$. ■

[35] Numerical analyses suggest that conditions (A.19) and (A.20) are satisfied for a large class of valuation distributions G_j. For instance, if $G_E(x_E) = x_E$, condition (A.19) holds for any G_I, and condition (A.20) holds for G_I that is relatively skewed to the left; i.e., the incumbent is "strong" in the sense that it is likely to have high valuation (see Figure 4). This includes the "power distributions" $G_I(x_I) = \left(\frac{x_I}{\overline{v}_I}\right)^3$ for all $\overline{v}_I \geq 3/2$.

A.9 Proof of Proposition 6

Consider the entrant's payoff:

$$\pi_E = \begin{cases} \tilde{a}_E - \max[b_I, R] + \delta \int_{a_I}^1 a_E - a_I \, d\tilde{F} & \text{if } b_E > \max[b_I, R], \\ 0 + \delta(\tilde{a}_E - a_I)^+ & \text{if } b_E \leq \max[b_I, R]. \end{cases}$$

The entrant's weakly dominant bid is $b_E^* = \tilde{a}_E + \delta \left(\int_0^1 (a_E - a_I)^+ \, d\tilde{F} - (\tilde{a}_E - a_I)^+ \right)$. Since $(a_E - a_I)^+$ is convex in a_E, Jensen's inequality implies that $\int_0^1 (a_E - a_I)^+ \, d\tilde{F} - (\tilde{a}_E - a_I)^+ \geq 0$; therefore, the entrant bids above its average Stage 1 per-impression valuation, $\tilde{a}_E$.

Finally, consider the incumbent's payoff:

$$\pi_I = \begin{cases} a_I - b_E + \delta(a_I - \tilde{a}_E)^+ & \text{if } b_I \geq \max[b_E, R], \\ 0 + \delta \int_0^1 (a_I - \max[a_E, R])^+ \, d\tilde{F} & \text{if } b_I < \max[b_E, R], b_E \geq R, \\ 0 + \delta(a_I - \tilde{a}_E)^+ & \text{if } b_I < \max[b_E, R], b_E < R. \end{cases} \tag{A.23}$$

Following the reasoning from the proof of Lemma 2 in Section A.3, we obtain that the incumbent's weakly dominant bid is

$$b_I^* = a_I + \delta \left((a_I - \tilde{a}_E)^+ - \int_0^1 (a_I - \max[a_E, R])^+ \, d\tilde{F} \right). \tag{A.24}$$

First, note that if $a_I \leq \tilde{a}_E$, then $b_I^* = a_I - \delta \int_0^1 (a_I - \max[a_E, R])^+ \, d\tilde{F} \leq a_I$; i.e., the incumbent underbids for low a_I. Second, consider $a_I > \tilde{a}_E$. We have that b_I^* is strictly increasing in a_I in this region because $\frac{\partial b_I^*}{\partial a_I} = \frac{\partial}{\partial a_I} \left(a_I + \delta \left(a_I - \tilde{a}_E - \int_0^1 (a_I - \max[a_E, R])^+ \, d\tilde{F} \right) \right) = 1 + \delta(1 - \tilde{F}(a_I)) > 0$.

Finally, $b_I^* > a_I$ at $a_I = 1$, because $\frac{1}{\delta}(b_I^* - a_I)$ is equal to $(1 - \tilde{a}_E) - \int_0^1 (1 - \max[a_E, R])^+ \, d\tilde{F} = 1 - \tilde{a}_E - \int_0^R 1 - R \, d\tilde{F} - \int_R^1 1 - a_E \, d\tilde{F} = 1 - \left(\int_0^R 1 + a_E - R \, d\tilde{F} + (1 - \tilde{F}(R)) \right) = \tilde{F}(R) - \int_0^R 1 + a_E - R \, d\tilde{F} \geq \tilde{F}(R) - (1 + R - R)\tilde{F}(R) = 0$. In sum, b_I^* is less than a_I at $a_I = \tilde{a}_E$, greater than a_I at $a_I = 1$, and strictly increasing in a_I. Therefore, by the Intermediate Value Theorem, there exists a unique root $\tilde{a} \in (\tilde{a}_E, 1)$ such that $b_I^* < a_I$ for all $a_I < \tilde{a}$, and $b_I^* > a_I$ for all $a_I > \tilde{a}$.

A.10 Proof of Proposition 7

We start by deriving the entrant's weakly dominant bid. Consider the entrant's payoff in Stage 1:

$$\pi_E = \begin{cases} \mu_E \left(1 - \frac{\max[\tilde{c}_I b_I, R]}{\mu_E}\right) + \delta \int_R^1 \left(\int_{c_I}^1 c_E - c_I \, dF_E\right) dF_I & \text{if } \mu_E b_E > \tilde{c}_I b_I, \\ 0 + \delta \int_R^1 (\mu_E - c_I)^+ \, dF_I & \text{if } \mu_E b_E \leq \tilde{c}_I b_I, \end{cases}$$

where $\tilde{c}_I$ is a random variable which follows F_I. Even if the entrant does not know the realization of $\tilde{c}_I$, its weakly dominant bid is

$$b_E^* = 1 + \frac{\delta}{\mu_E} \int_R^1 \left(\int_0^1 (c_E - c_I)^+ \, dF_E\right) - (\mu_E - c_I)^+ \, dF_I. \tag{A.25}$$

And since $(c_E - c_I)^+$ is convex in c_E for all realizations of c_I, Jensen's inequality implies that $\int_0^1 (c_E - c_I)^+ \, dF_E - (\mu_E - c_I)^+ \geq 0$ for all c_I. Hence, $b_E^* \geq 1$; i.e., the entrant overbids.

A.11 Proof of Proposition 8

Proof. Following the argument in the main model, whether the incumbent bids below or above valuation depends on the sign of $\int_{x \in \mathcal{X}} \left((c_I - \max[\mu_x, R])^+ - \int_0^1 (c_I - \max[c_E, R])^+ \, dF_x\right) dP(x)$. But we have shown in the main model that for any distribution F_x of c_E, there exists a pair of thresholds $(\tilde{c}_1(x), \tilde{c}_2(x))$ such that the incumbent underbids for all $c_I < \tilde{c}_1(x)$ and overbids for all $c_I > \tilde{c}_2(x)$ (see proof of Proposition 2 in Section A.4). It follows that the integral above is negative for all $c_I < \underline{c}' \equiv \inf_{x \in \mathcal{X}} \tilde{c}_1(x)$ and positive for all $c_I > \overline{c}' \equiv \sup_{x \in \mathcal{X}} \tilde{c}_2(x)$. This completes the proof. ■

A.12 Proof of Proposition 9

Proof. From (A.1), we have

$$\mathbb{E}[\pi_E] = \begin{cases} \mu_E \left(1 - \frac{\max[c_I b_{I1}, R]}{\mu_E}\right) + \delta \int_{c_I}^1 c_E \left(1 - \frac{c_I}{c_E}\right) dF_E + \alpha & \text{if } b_{E1} > \frac{\max[c_I b_{I1}, R]}{\mu_E}, \\ \delta \mu_E \left(1 - \frac{c_I}{\mu_E}\right)^+ & \text{if } b_{E1} \leq \frac{\max[c_I b_{I1}, R]}{\mu_E}. \end{cases}$$

Thus, a weakly dominant Stage 1 bid is $b_{E1}^*(\alpha) = 1 + \frac{\delta}{\mu_E} \left(\int_{c_I}^1 c_E - c_I \, dF_E - (\mu_E - c_I)^+\right) + \frac{\alpha}{\mu_E}$. ■

A.13 Proof of Proposition 10

Proof. First, we establish that from the publisher's profit perspective, offering ad credit α is equivalent to artificially increasing the entrant's effective bid by α.

Consider the latter mechanism. The Stage 2 outcomes are identical for both cases. In Stage 1, the advertisers' weakly dominants can be easily verified to be the same as the main model; i.e., the entrant bids b^*_{E1} as in (3.2) and the incumbent bids b^*_{I1} as in (3.1). Note that this implies that the effective bids are the same as the former mechanism where the publisher gives free ad credit α if the entrant wins. To see this, under the former mechanism, the entrant's effective bid is $\mu_E b^*_{E1}(\alpha)$ as in (5.1). But from (5.1), we have $\mu_E b^*_{E1}(\alpha) = \mu_E b^*_{E1} + \alpha$, which is equivalent to entrant's effective bid under the artificial additive boosting mechanism. The incumbent's effective bids are trivially the same. Therefore, the mechanisms have the same allocation rule.

Moreover, the payoffs of the two mechanisms are identical. If the incumbent wins in Stage 1, the publisher's Stage 1 profit is $\mu_E b^*_{E1} + \alpha$ in both cases, and if the entrant wins, it is $\max[c_I b^*_{I1}] - \alpha$ in both cases. In sum, the two mechanism yield the same profit for the publisher.

Now, consider the boosting multiplier β. For any given additive term α, if the publisher sets $\beta(\alpha) = 1 + \frac{\alpha}{\mu_E b^*_{E1}}$, then the advertisers' effective bids are the same as in the mechanism wherein the publisher adds α to the entrant's effective bid. Therefore, the two mechanisms have the same allocation rule.

Furthermore, note that the advertisers' bids are the same for both mechanisms: the incumbent (entrant) bids b^*_{I1} (b^*_{E1}). This means that both mechanisms can be cast as a "direct mechanism" where advertisers bid their true "type." Therefore, by the Revenue Equivalence principle (Myerson, 1981), the two mechanisms yield the same expected profit up to a constant. ∎

A.14 Proof of Proposition 11

Proof. From the derivation of the LREF Stage 1 bids (see Section OA2 of the online appendix), we see that an important condition that shapes the outcome of the auction is

$$\delta \left(\mathbb{E}[\pi^l_E] - \left(\mathbb{E}[\pi^0_i] - \mathbb{E}[\pi^l_i] \right) \right) \leq \theta(c_i - \mu_E). \tag{A.26}$$

We can write this condition in terms of the reserve price R. First, note that $\mathbb{E}[\pi_E^l] = \int_{c_i}^{c_I} \theta(c_E - c_i)\, dF_E + \int_{c_i}^{c_I} (c_E - c_I) + \theta(c_I - c_i)\, dF_E$ is independent of R, and

$$\begin{aligned}\mathbb{E}[\pi_i^0] - \mathbb{E}[\pi_i^l] &= c_I - \mu_E - \int_0^1 (c_i - \max[c_E, R])^+ \; dF_E \\ &= \int_0^1 c_I - c_E\, dF_E - \int_0^1 (c_i - \max[c_E, R])^+ \; dF_E \\ &= \int_0^R R - c_E\, dF_E - \int_{c_i}^1 c_E - c_i\, dF_E. \end{aligned} \tag{A.27}$$

Second, since (A.27) is strictly increasing in R, we obtain that (A.26) is equivalent to $R \geq \hat{R}$, where $\hat{R}$ solves $\delta\left(\mathbb{E}[\pi_E^l] - \left(\mathbb{E}[\pi_i^0] - \mathbb{E}[\pi_i^l]\right)\right) = \theta(c_i - \mu_E)$.

Case 1: Suppose $\mu_E \leq c_i < c_I$. Following the derivation of LREF bids in Section OA2.2, if $R > \hat{R}$, then the LREF equilibrium is $[I, i, e]$ and the entrant bids $b_E^*(\delta) = 1 + \frac{\delta}{\theta \mu_E}\mathbb{E}[\pi_E^l]$. Therefore, $b_E^*(\delta) \geq b_E^*(0)$ for all $\delta > 0$. For the weak incumbent, we obtain $b_i^*(\delta) = 1 - \theta + \frac{\theta \mu_E}{c_i} b_E^*(\delta)$; therefore $b_i^*(\delta) > b_i^*(0)$ for all $\delta > 0$.

If $R < \hat{R}$, then the LREF equilibrium is $[I, E, i]$ and $b_E^*(\delta) = 1 - \theta + \frac{\theta}{\mu_E} \max[c_i b_i^*(\delta), R]$ and $b_i^*(\delta) = \left(1 + \frac{\delta}{\theta c_i}\left(\mathbb{E}[\pi_i^0] - \mathbb{E}[\pi_i^l]\right)\right)^+$. To determine whether the advertisers bid below or above their learning-free benchmarks, we need to determine the sign of $\mathbb{E}[\pi_i^0] - \mathbb{E}[\pi_i^l]$. Since (i) $\frac{\partial (A.27)}{\partial R} > 0$, (ii) $(A.27) = -\int_{c_i}^1 c_E - c_i\, dF_E < 0$ at $R = 0$, and (iii) $(A.27) = c_i - \mu_E > 0$ at $R = c_i$, there exists a unique $\tilde{R} \in (0, c_i)$ such that $\mathbb{E}[\pi_i^0] - \mathbb{E}[\pi_i^l] < 0$ for all $R < \tilde{R}$ and $\mathbb{E}[\pi_i^0] - \mathbb{E}[\pi_i^l] > 0$ for all $R > \tilde{R}$. Thus, $R < \tilde{R} \Rightarrow b_E^*(\delta) \leq b_E^*(0)$ and $b_i^*(\delta) \leq b_i^*(0)$ for all $\delta > 0$; and $R > \tilde{R} \Rightarrow \mathbb{E}[\pi_i^0] - \mathbb{E}[\pi_i^l] > 0$, so that $b_E^*(\delta) \geq b_E^*(0)$ and $b_i^*(\delta) \geq b_i^*(0)$ for all $\delta > 0$.

Case 2: Suppose $c_i < \mu_E \leq c_I$. The LREF equilibrium is $[I, E, i]$.

The EF conditions for the strong incumbent and the entrant are $c_I(1 - \max[\mu_E b_E, R]/c_I) + \delta\mathbb{E}[\pi_i^l] \geq \theta c_I(1 - \max[c_i b_i, R]/c_I) + \delta\mathbb{E}[\pi_i^l] \iff \mu_E b_E \leq (1-\theta)c_I + \theta \max[c_i b_i, R]$ and $\mu_E(1 - \max[\mu_E b_E, R]/\mu_E) + \delta\mathbb{E}[\pi_E^l] \leq \theta\mu_E(1 - \max[c_i b_i, R]/\mu_E) + \delta\mathbb{E}[\pi_E^l] \iff \mu_E b_E \geq (1-\theta)\mu_E + \theta \max[c_i b_i, R]$. Therefore, the entrant's LREF bid is $b_E^*(\delta) = 1 - \theta + \frac{\theta}{\mu_E}\max[c_i b_i^*, R]$.

The EF conditions for the the weak incumbent and the entrant are $\theta c_i(1 - \max[c_i b_i, R]/c_i) + \delta\mathbb{E}[\pi_i^0] \leq \delta\mathbb{E}[\pi_i^l] \iff \theta c_i - \delta\mathbb{E}[\pi_i^l] \leq \theta \max[c_i b_i, R]$ and $\theta\mu_E(1 - \max[c_i b_i, R]/\mu_E) + \delta\mathbb{E}[\pi_E^l] \geq \delta\mathbb{E}[\pi_E^0] \iff \theta \max[c_i b_i, R] \leq \theta\mu_E + \delta(\mathbb{E}[\pi_E^l] - \mathbb{E}[\pi_E^0])$. Since the LREF bid binds at the lower-bound, we have $\max[c_i b_i^*, R] = c_i - \frac{\delta}{\theta}\mathbb{E}[\pi_i^l]$. Thus, $R < c_i - \frac{\delta}{\theta}\mathbb{E}[\pi_i^l] \Rightarrow b_i^*(\delta) = 1 - \frac{\delta}{\theta c_i}\mathbb{E}[\pi_i^l]$ and

$b^*_E(\delta) = 1-\theta+\frac{\theta}{\mu_E}\left(c_i - \frac{\delta}{\theta}\mathbb{E}[\pi^l_i]\right)$, such that $b^*_E(\delta) \leq b^*_E(0)$ and $b^*_i(\delta) \leq b^*_i(0)$ for all $\delta > 0$.

On the other hand, if $R \geq c_i - \frac{\delta}{\theta}\mathbb{E}[\pi^l_i]$, then the lowest bid b_i that satisfies the EF condition, $c_i - \frac{\delta}{\theta}\mathbb{E}[\pi^l_i] \leq \max[c_i b_i, R] \leq \mu_E + \frac{\delta}{\theta}(\mathbb{E}[\pi^l_E] - \mathbb{E}[\pi^0_E])$, is $b^*_i = 0$. In this case, the LREF bids are $b^*_E(\delta) = 1-\theta+\frac{\theta}{\mu_E}R$ and $b^*_I(\delta) = 0$, such that the bids are the same across environments with and without learning. For this to hold for all $\delta > 0$, we must have $R \geq c_i$.

Taken together, we can define the threshold for the reserve price R above which the stated result holds: $\overline{R}$ equals $\min\left[\hat{R}, \tilde{R}\right]^+$ if $\mu_E \leq c_i$, and equals c_i otherwise. ■

A.15 Proof of Lemma 4

Entrant

Proof. If $\mu_E \leq c_i$, then the entrant clearly wants to reveal its CTR, for otherwise, it would have no chance of winning in the Stage 2. If $c_i < \mu_E \leq c_I$, then the difference between its payoffs when its CTR is revealed and concealed is

$$\begin{aligned}
\Delta_1 &\equiv \int_{c_i}^{c_I} \theta(c_E - c_i)\,dF_E + \int_{c_I}^{1} c_E - c_I + \theta(c_I - c_i)\,dF_E - \theta(\mu_E - c_i)\\
&= -\int_0^{c_i} \theta(c_E - c_i)\,dF_E + \int_0^{c_i} \theta(c_E - c_i)\,dF_E + \int_{c_i}^{c_I} \theta(c_E - c_i)\,dF_E + \int_{c_I}^{1} c_E - c_I + \theta(c_I - c_i)\,dF_E\\
&\quad - \theta(\mu_E - c_i)\\
&= -\int_0^{c_i} \theta(c_E - c_i)\,dF_E + \int_0^{c_i} \theta(c_E - c_i) - \theta(c_E - c_i)\,dF_E + \int_{c_i}^{c_I} \theta(c_E - c_i) - \theta(c_E - c_i)\,dF_E\\
&\quad + \int_{c_I}^{1} c_E - c_I + \theta(c_I - c_i) - \theta(c_E - c_i)\,dF_E\\
&= \int_0^{c_i} \theta(c_i - c_E)\,dF_E + \int_{c_I}^{1} (1-\theta)(c_E - c_I)\,dF_E > 0.
\end{aligned} \tag{A.28}$$

Therefore, the entrant is better off revealing its CTR when $c_i < \mu_E \leq c_I$.

Finally, if $c_I < \mu_E$, then the difference between its payoffs when its CTR is revealed and concealed is $\Delta_2 \equiv \int_{c_i}^{c_I} \theta(c_E - c_i)\,dF_E + \int_{c_I}^{1}(c_E - c_I) + \theta(c_I - c_i)\,dF_E - ((\mu_E - c_I) + \theta(c_I - c_i))$, and differentiating Δ_2 with respect to θ yields $\frac{\partial \Delta_2}{\partial \theta} = c_i - c_I + \int_{c_i}^{c_I} c_E - c_I\,dF_E - \int_{c_I}^{1} c_i - c_I\,dF_E = \int_0^{c_i} c_i - c_I\,dF_E + \int_{c_i}^{c_I} c_E - c_I\,dF_E < 0$. Since Δ_2 is decreasing in θ, it suffices to show that $\Delta_2 > 0$ at $\theta = 1$ to prove that $\Delta_2 > 0$ for all $\theta \in (0,1)$. But the positivity holds at $\theta = 1$ because $\Delta_2|_{\theta=1} = \int_{c_i}^{c_I} c_E - c_I\,dF_E + \int_{c_I}^{1} c_E - c_I + c_I - c_i dF_E - \int_0^1 c_E -$

$c_I + c_I - c_I \, dF_E = \int_0^{c_i} c_i - c_E \, dF_E > 0$. This completes the proof. ∎

Incumbent with Lower CTR

Proof. First, if $\mu_E > c_i$, then it is clearly better for the incumbent with the lower CTR if the entrant's CTR is learned because *only then* does it have a chance of winning the second ad slot. Second, suppose $\mu_E \leq c_i$. The difference between its payoffs when the entrant's CTR is revealed and concealed is $\Delta_3 \equiv \int_0^{c_i} \theta(c_i - c_E) \, dF_E - \theta(c_i - \mu_E) = -\int_{c_i}^1 \theta(c_i - c_E) \, dF_E \geq 0$. This completes the proof. ∎

Incumbent with Higher CTR

Proof. Suppose $\mu_E \leq c_i$. Then the difference between the payoffs of the incumbent with the higher CTR when the entrant's CTR is revealed and when it is not is $\Delta_4 \equiv F_E(c_i)(c_I - c_i) + \int_0^{c_i} \theta(c_i - c_E) \, dF_E + \int_{c_i}^{c_I} (c_I - c_E) + \theta(c_E - c_i) \, dF_E + (1 - F_E(c_I))\theta(c_I - c_i) - ((c_I - c_i) + \theta(c_i - \mu_E))$. Differentiating Δ_4 with respect to θ yields $\frac{\partial \Delta_4}{\partial \theta} = -c_i + \int_0^{c_i} c_i - c_E \, dF_E + \int_{c_i}^{c_I} c_E - c_I \, dF_E + \int_{c_I}^1 c_I - c_I \, dF_E + \mu_E = \int_0^{c_i} c_i - c_E - c_i + c_E \, dF_E + \int_{c_i}^{c_I} c_E - c_i - c_i + c_E \, dF_E + \int_{c_I}^1 c_I - c_i - c_i + c_E \, dF_E = 2 \int_{c_i}^{c_I} c_E - c_I \, dF_E + \int_{c_I}^1 c_I - c_I \, dF_E + \int_{c_I}^1 c_E - c_I \, dF_E > 0$, where the last positivity follows from the positivity of each term in the sum in the line before. Therefore, Δ_4 is strictly increasing in θ. At the end points of θ, we have $\Delta_4 < 0$ and $\Delta_4 > 0$, respectively: $\Delta_4|_{\theta=0} = c_i - c_I + \int_0^{c_i} c_I - c_I \, dF_E + \int_{c_i}^{c_I} c_I - c_E \, dF_E = \int_{c_i}^{c_I} c_i - c_E \, dF_E + \int_{c_I}^1 c_i - c_I \, dF_E < 0$, and $\Delta_4|_{\theta=1} = \mu_E - c_I + \int_0^{c_i} c_I - c_I \, dF_E + \int_{c_I}^1 c_I - c_I \, dF_E + \int_0^{c_i} c_i - c_E \, dF_E + \int_{c_i}^{c_I} c_I - c_I \, dF_E = \int_{c_I}^1 c_E - c_I \, dF_E + \int_{c_i}^{c_I} c_E - c_I \, dF_E > 0$. Therefore, by the IVT, there exists a unique $\hat{\theta} \in (0, 1)$ that solves $\Delta_4(\theta) = 0$ such that $\Delta_4 < 0$ for $\theta < \hat{\theta}$ and $\Delta_4 > 0$ for $\theta > \hat{\theta}$. This means that the incumbent with the higher CTR wants the entrant's CTR to be learned if and only if the position-CTR of the second slot is sufficiently large.

Next, suppose $c_i < \mu_E \leq c_I$. Then the difference between the payoffs of the incumbent with the higher CTR when the entrant's CTR is revealed and when it is not is $\Delta_5 \equiv F_E(c_i)(c_I - c_i) + \int_0^{c_i} \theta(c_i - c_E) \, dF_E + \int_{c_i}^{c_I} (c_I - c_E) + \theta(c_E - c_i) \, dF_E + (1 - F_E(c_I))\theta(c_I - c_i) - ((c_I - \mu_E) + \theta(\mu_E - c_i))$. The difference Δ_5 strictly decreases with c_I because $\frac{\partial \Delta_5}{\partial c_I} = -1 + F_E(c_I) + \theta(1 - F_E(c_I)) = -(1 - \theta)(1 - F_E(c_I)) < 0$. Next, we examine the sign of Δ_5 at the endpoints $c_I = \mu_E$ and $c_I = 1$. At $c_I = 1$, we have $\Delta_5|_{c_I=1} = \int_0^{c_i} 1 - c_i + \theta(c_i - c_E) \, dF_E + \int_{c_i}^1 1 - c_E + \theta(c_E - c_i) \, dF_E - ((1 - \mu_E) + \theta(\mu_E - c_i)) = (2\theta - 1) \int_0^{c_i} c_i - c_E \, dF_E$. Therefore, $\Delta_5|_{c_I=1}$ is positive if $\theta \geq \frac{1}{2}$ and negative otherwise. And since Δ_5 is decreasing in c_I, we obtain $\Delta_5 \geq 0$ for all c_I if $\theta \geq \frac{1}{2}$.

The sign of Δ_5 is only inconclusive for $\theta < \frac{1}{2}$, so we examine this case next. If $\Delta_5|_{c_I=\mu_E} \leq 0$ under $\theta < \frac{1}{2}$, then by the decreasing property of Δ_5 we would obtain $\Delta_5 \leq 0$ for all c_I. On the other hand, if $\Delta_5|_{c_I=\mu_E} > 0$, then by the IVT, there would exist a unique zero $\tilde{c}_I$ such that $\Delta_5 > 0$ for all $c_I < \tilde{c}_I$, and $\Delta_5 < 0$ for all $c_I > \tilde{c}_I$. We will show that the latter holds; i.e., $\Delta_5|_{c_I=\mu_E} > 0$:

$$\begin{aligned}\Delta_5|_{c_I=\mu_E} &= \int_0^{c_i} \mu_E - c_i + \theta(c_i - c_E)\, dF_E + \int_{c_i}^{\mu_E} \mu_E - c_E + \theta(c_E - c_i)\, dF_E + \int_{\mu_E}^1 \theta(\mu_E - c_i)\, dF_E \\ &\quad - \theta(\mu_E - c_i) \\ &= \int_0^{c_i} (2\theta - 1)(c_i - c_E)\, dF_E + \int_{\mu_E}^1 (1-\theta)(c_E - \mu_E)\, dF_E.\end{aligned}$$

Since this is a linear function of θ, to show that $\Delta_5|_{c_I=\mu_E} > 0$ for all $\theta \in \left(0, \frac{1}{2}\right)$, it suffices to show that $\Delta_5|_{c_I=\mu_E} > 0$ at the endpoints $\theta = 0$ and $\theta = \frac{1}{2}$. At $\theta = \frac{1}{2}$, we have $\Delta_5|_{c_I=\mu_E} = \frac{1}{2} \int_{\mu_E}^1 c_E - \mu_E\, dF_E > 0$. At $\theta = 0$, we have $\Delta_5|_{c_I=\mu_E}(\theta = 0) = \int_{\mu_E}^1 c_E - \mu_E\, dF_E - \int_0^{c_i} c_i - c_E\, dF_E = \int_0^{c_i} \mu_E - c_I\, dF_E + \int_{c_i}^{\mu_E} \mu_E - c_E\, dF_E > 0$, where the last positivity follows from $\mu_E > c_i$. Thus, we have shown that $\Delta_5|_{c_I=\mu_E} > 0$ for all $\theta < \frac{1}{2}$. Combined with the fact that $\Delta_5|_{c_I=1} < 0$ for $\theta < \frac{1}{2}$, the IVT implies the unique existence of $\tilde{c}_I \in (\mu_E, 1)$ such that $\Delta_5 > 0$ for all $c_I < \tilde{c}_I$, and $\Delta_5 < 0$ for all $c_I > \tilde{c}_I$.

In summary, if $c_i < \mu_E \leq c_I$, then Δ_5 is negative if $\theta < \frac{1}{2}$ and $c_I > \tilde{c}_I$, and it is positive otherwise. We consider the last case of $c_I < \mu_E$. The difference between the payoffs of the incumbent with the higher CTR when the entrant's CTR is revealed and when it is not is $\Delta_6 \equiv F_E(c_i)(c_I - c_i) + \int_0^{c_i} \theta(c_i - c_E)\, dF_E + \int_{c_i}^{c_I} (c_I - c_E) + \theta(c_E - c_i)\, dF_E + (1 - F_E(c_I))\theta(c_I - c_i) - \theta(c_I - c_i)$. Differentiating Δ_6 with respect to c_I yields $\frac{\partial \Delta_6}{\partial c_I} = F_E(c_I) - \theta F_E(c_I) = (1-\theta)F_E(c_I) > 0$. Since Δ_6 is increasing in c_I, to show $\Delta_6 > 0$ for all c_I, it suffices to show positivity only at the lowest value $c_I = c_i$. But if $c_I = c_i$, then Δ_6 simplifies to $\int_0^{c_i} \theta(c_i - c_E)\, dF_E$, which is clearly positive. Therefore, $\Delta_6 > 0$ if $c_I < \mu_E$. ■

Online Appendix to "Learning in Online Advertising"

OA1 Extended Discussion of Ad Credit

The following proposition states the condition under which offering ad credit is profitable for the publisher.

Proposition OA1. *If the probability of the incumbent's CTR being high, $\mathbb{P}\{c_I > \tilde{c}\}$ (where $\tilde{c}$ is defined in the proof), is sufficiently high, then offering ad credit increases the publisher's revenue.*

Proof. We will derive a condition equivalent to $\frac{\partial}{\partial \alpha}\mathbb{E}[\pi_P(\alpha)]|_{\alpha=0} > 0$, which is a sufficient condition for the publisher to provide positive ad credit.

First, to simplify notation, let $\tilde{b}_E \equiv \mu_E + \delta\left(\int_{c_I}^1 (c_E - c_I)\, dF_E - (\mu_E - c_I)^+\right)$, so that the entrant's first stage effective bid is $\mu_E b^*_{E1} = \alpha + \tilde{b}_E$. Based on this, denote the terms in the publisher's revenue expression that are independent of α by $\xi_1 \equiv \tilde{b}_E + \delta \max[\min[c_I, \mu_E], R]$ and $\xi_2 \equiv \max[c_I b^*_{I1}, R] + \delta\left(\int_0^{c_I} \max[c_E, R]\, dF_E + (1 - F_E(c_I))c_I\right)$. Then, we have $\Pi_P(\alpha, c_I)$ equal to $\alpha + \xi_1$ if $c_I \geq \min[\hat{c}(\alpha), 1]$, and equal to $-\alpha + \xi_2$ otherwise, where $\hat{c}(\alpha)$ is the value such that $c_I b^*_{I1} \geq \mu_E b^*_{E1}(\alpha)$ for all $c_I \geq \hat{c}(\alpha)$, and $c_I b^*_{I1} < \mu_E b^*_{E1}(\alpha)$ for all $c_I < \hat{c}(\alpha)$. Note that we can show the unique existence of such threshold following a similar reasoning as in the first part of the proof of Proposition 3. Using these notations, we can re-write the publisher's revenue as follows: $\mathbb{E}[\pi_P(\alpha)] = \int_0^1 \Pi_P(\alpha, c_I)\, dF_I = \int_0^{\min[\hat{c}(\alpha),1]} -\alpha + \xi_2\, dF_I + \int_{\min[\hat{c}(\alpha),1]}^1 \alpha + \xi_1\, dF_I$. Therefore, if $\hat{c}(\alpha) > 1$, which occurs if and only if $\alpha > 1 - \mu_E + \delta \int_0^1 \max[c_E, R] - c_E\, dF_E$, then the derivative with respect to α is $\frac{\partial}{\partial \alpha}\mathbb{E}[\pi_P(\alpha)] = \frac{\partial}{\partial \alpha}\int_0^1 -\alpha + \xi_2\, dF_I = \frac{\partial}{\partial \alpha}(-\alpha) = -1$. On the other hand, if $\hat{c}(\alpha) \leq 1$, then $\frac{\partial}{\partial \alpha}\mathbb{E}[\pi_P(\alpha)] = \frac{\partial}{\partial \alpha}\int_0^{\hat{c}(\alpha)} -\alpha + \xi_2\, dF_I + \frac{\partial}{\partial \alpha}\int_{\hat{c}(\alpha)}^1 \alpha + \xi_1\, dF_I$ $= f_I(\hat{c}(\alpha))\, \hat{c}'(\alpha)(-2\alpha - \xi_1 + \xi_2) + 1 - 2F_I(\hat{c}(\alpha))$. Rearranging terms, we obtain that $\frac{\partial}{\partial \alpha}\mathbb{E}[\pi_P(\alpha)] > 0$ if and only if

$$\alpha \leq 1 - \mu_E + \delta \int_0^1 \max[c_E, R] - c_E\, dF_E \tag{OA1}$$

and

$$\mathbb{P}\{c_I > \hat{c}(\alpha)\} > \frac{1 - f_I(\hat{c}(\alpha))\, \hat{c}'(\alpha)(-2\alpha - \xi_1 + \xi_2)}{2}. \tag{OA2}$$

Substituting $\alpha = 0$, condition OA1 holds and condition OA2 simplifies as stated in the proposition with $\tilde{c} \equiv \hat{c}(0)$. ■

An important implication of Proposition OA1 is that the publisher should consider offering ad credit if the incumbent's CTR is likely to be high. The reason is that if the incumbent is strong, the positive effects associated with extraction and learning effects are large while the negative cost effect is small. Put together, the three forces create strong incentives for the publisher to provide ad credit to new advertisers. Next, we characterize the three effects as a function of the ad credit level.

Three Forces

First, the extraction effect is defined as the expected increase in the publisher's Stage 1 payoff due to the ad credit when the incumbent wins: $\int_R^1 (\mu_E b_{E1}^*(\alpha) - \mu_E b_{E1}^*(0)) \mathbb{I}_{\{c_I b_{I1}^* \geq \mu_E b_{E1}^*(\alpha)\}} \, dF_I$, which simplifies to $\alpha \, \mathbb{P}\{c_I b_{I1}^* \geq \mu_E b_{E1}^*(\alpha)\}$. Second, the learning effect is defined as the additional Stage 2 revenue the publisher gains from offering ad credit, compared to the Stage 2 payoff when it does not offer any; i.e., $\delta\left(\pi_{P2}(\alpha) - \pi_{P2}(0)\right)$ where

$$\pi_{P2}(\alpha) = \int_{c_I b_{I1}^* \geq \mu_E b_{E1}^*(\alpha)} \min[c_I, \mu_E] \, dF_I + \int_{c_I b_{I1}^* < \mu_E b_{E1}^*(\alpha)} \left(\int_0^{c_I} \max[c_E, R] \, dF_E + (1 - F_E(c_I)) c_I \right) dF_I.$$

Proposition OA2 (Three Forces Generated by Ad Credit)**.** *Suppose c_I and c_E are distributed according to $F_I(c) = c^n$ and $F_E(c) = c$, respectively, for $c \in [0,1]$, and $R \geq \frac{\sqrt{1+\delta}-1}{\delta}$. Then the extraction effect is concave in α, and the expected cost of ad credit increases monotonically in α. The learning effect increases convexly in α for $\alpha \leq \frac{1+R^2\delta}{2}$, and plateaus thereafter.*

Proof. Extraction effect: From the proof of Proposition 3, we obtained the unique existence of $\hat{c} > \mu_E = \frac{1}{2}$ such that the entrant wins the first stage auction for all $c_I < \hat{c}$ and the incumbent wins for all $c_I \geq \hat{c}$. Since this result holds without ad credit (i.e., $\alpha = 0$), it must be that when the publisher offers positive ad credit to the entrant, the threshold c_I after which the incumbent wins must be at least as large as $\hat{c}$. For otherwise, it would imply that the incumbent wins for a larger interval of c_I even if the ad credit makes the entrant's ad score more competitive (see Proposition 9), which cannot be true. Therefore, we obtain that if the publisher offers non-zero ad credit, the entrant wins the first stage auction for all $c_I < \mu_E = \frac{1}{2}$. This property helps simplify the expression for the two positive forces generated by the ad credit. The extraction effect can be written as

$$\begin{aligned} \int_0^1 (\mu_E b_{E1}^*(\alpha) - \mu_E b_{E1}^*(0)) \mathbb{I}_{\{c_I b_{I1}^* \geq \mu_E b_{E1}^*(\alpha)\}} \, dF_I &= \alpha \int_{c_I b_{I1}^* \geq \mu_E b_{E1}^*(\alpha)} dF_I \\ &= \alpha \int_{c_I \geq \frac{1}{2}, c_I b_{I1}^* \geq \mu_E b_{E1}^*(\alpha)} n c_I^{n-1} \, dc_I \qquad \text{(OA3)} \end{aligned}$$

Next, we will express the set $\{c_I : c_I \geq \frac{1}{2}, c_I b_{I1}^* \geq \mu_E b_{E1}^*(\alpha)\}$ in terms of c_I. To that end, we have $c_I b_{I1}^* = c_I \left(1 + \delta \left(\left(1 - \frac{\mu_E}{c_I}\right)^+ - \int_0^1 \left(1 - \frac{\max[c_E, R]}{c_I}\right)^+ dF_E \right) \right) = c_I + \delta \left(c_I - \frac{1}{2} + \frac{R^2 - c_I^2}{2}\right)$ and $\mu_E b_{E1}^*(\alpha) = \mu_E \left(1 + \frac{\alpha}{\mu_E} + \frac{\delta}{\mu_E} \left(\int_{c_I}^1 (c_E - c_I) \, dF_E - (\mu_E - c_I)^+ \right) \right) = \frac{1}{2} + \alpha + \frac{\delta}{2}(1 - c_I)^2$, which implies that the condition "$c_I \geq \frac{1}{2}$ and $c_I b^*_{I1} \geq \mu_E b^*_{E1}(\alpha)$" is equivalent to

$$\alpha \leq \frac{2\delta + 2\delta^2 R^2 + 1}{4\delta} \text{ and } c_I \geq \max\left(\frac{1}{2}, \frac{2\delta - \sqrt{-4\alpha\delta + 2\delta + 2\delta^2 R^2 + 1} + 1}{2\delta}\right) \equiv \tilde{c}_3.$$

Therefore, we have

$$(OA3) = \begin{cases} \alpha(1 - F_I(\tilde{c}_3)) & \text{if } \alpha \leq \frac{2\delta+2\delta^2R^2+1}{4\delta}, \\ 0 & \text{if } \alpha > \frac{2\delta+2\delta^2R^2+1}{4\delta}. \end{cases}$$

Finally, to show that $(OA3)$ is concave in α for $\alpha \leq \frac{2\delta+2\delta^2R^2+1}{4\delta}$, it suffices to establish that $1 - F_I(\tilde{c}_3)$ is decreasing and concave in α. For then, $\frac{\partial^2 \alpha(1-F_I(\tilde{c}_3))}{\partial \alpha^2} = 2(1 - F_I(\tilde{c}_3))' + \alpha(1 - F_I(\tilde{c}_3))'' < 0$. But the decreasing property and concavity hold because $(1-F_I(\tilde{c}_3))' = (1-(\tilde{c}_3)^n)' = -\frac{n\left(\frac{2\delta-\sqrt{-4\alpha\delta+2\delta+2\delta^2R^2+1}+1}{2\delta}\right)^{n-1}}{\sqrt{-4\alpha\delta+2\delta+2\delta^2R^2+1}} < 0$ and $(1 - F_I(\tilde{c}_3))'' = (1 - (\tilde{c}_3)^n)''$, which in turn equals to

$$\begin{aligned}
&-\frac{n4\delta^2\left(\frac{q_1(\alpha,\delta,n,R)}{2\delta}\right)^n q_2(\alpha,\delta,n,R)}{(-4\alpha\delta + 2\delta + 2\delta^2R^2 + 1)^{3/2}\, q_1(\alpha,\delta,n,R)^2} \\
&\propto -q_1(\alpha,\delta,n,R)^n q_2(\alpha,\delta,n,R) \\
&= -q_1(\alpha,\delta,n,R)^{n+1} - q_1(\alpha,\delta,n,R)^n (n-1)\sqrt{-4\alpha\delta + 2\delta + 2\delta^2R^2 + 1} \\
&\leq -q_1(\alpha,\delta,n,R)^{n+1} < 0,
\end{aligned}$$

where

$$\begin{aligned}
q_1(\alpha,\delta,n,R) &= 2\delta - \sqrt{-4\alpha\delta + 2\delta + 2\delta^2R^2 + 1} + 1, \\
q_2(\alpha,\delta,n,R) &= 2\delta + n\sqrt{-4\alpha\delta + 2\delta + 2\delta^2R^2 + 1} - 2\sqrt{-4\alpha\delta + 2\delta + 2\delta^2R^2 + 1} + 1.
\end{aligned}$$

The last inequality $-q_1(\alpha,\delta,n,R)^{n+1} < 0$ follows from

$$\begin{aligned}
-\left(2\delta - \sqrt{-4\alpha\delta + 2\delta + 2\delta^2R^2 + 1} + 1\right)^{n+1} < 0 &\impliedby 2\delta - \sqrt{-4\alpha\delta + 2\delta + 2\delta^2R^2 + 1} + 1 > 0 \\
&\iff \alpha > -\frac{1}{2} - \delta\left(1 - \frac{R^2}{2}\right) \\
&\impliedby 0 > -\frac{1}{2} - \delta\left(1 - \frac{R^2}{2}\right) \\
&\impliedby \delta \geq 0, R \leq 1.
\end{aligned}$$

Expected cost: Recall that the entrant wins in Stage 1 for $c_I \leq \frac{1}{2}$, so the expected ad cost is increasing in α for this region. We will show that the expected cost for $c_I > \frac{1}{2}$ is also increasing in α. To that end, it suffices to show that

$$\frac{\partial}{\partial \alpha}\int_{c_I\geq\frac{1}{2},c_I b^*_{I1}\geq\mu_E b^*_{E1}(\alpha)} dF_I \leq 0. \tag{OA4}$$

For then,

$$\begin{aligned}\frac{\partial}{\partial \alpha}\alpha\int_{c_I\geq\frac{1}{2},c_I b^*_{I1}<\mu_E b^*_{E1}(\alpha)} dF_I &= \frac{\partial}{\partial \alpha}\alpha\left(1-\int_{c_I\geq\frac{1}{2},c_I b^*_{I1}\geq\mu_E b^*_{E1}(\alpha)} dF_I\right)\\ &= \underbrace{1-\int_{c_I\geq\frac{1}{2},c_I b^*_{I1}\geq\mu_E b^*_{E1}(\alpha)} dF_I}_{+\text{ by probability measure}} \underbrace{-\alpha\frac{\partial}{\partial \alpha}\int_{c_I\geq\frac{1}{2},c_I b^*_{I1}\geq\mu_E b^*_{E1}(\alpha)} dF_I}_{-\text{ by hypothesis}} \geq 0.\end{aligned}$$

But the desired inequality (OA4) holds because

$$\begin{aligned}\frac{\partial}{\partial \alpha}\int_{c_I\geq\frac{1}{2},c_I b^*_{I1}\geq\mu_E b^*_{E1}(\alpha)} dF_I &= \frac{\partial}{\partial \alpha}\begin{cases}1-F_I(\tilde{c}_3) & \text{if } \alpha\leq\frac{2\delta+2\delta^2R^2+1}{4\delta},\\ 0 & \text{if } \alpha>\frac{2\delta+2\delta^2R^2+1}{4\delta}\end{cases}\\ &= \begin{cases}-f_I(\tilde{c}_3)\frac{\partial\tilde{c}_3}{\partial\alpha} & \text{if } \alpha\leq\frac{2\delta+2\delta^2R^2+1}{4\delta},\\ 0 & \text{if } \alpha>\frac{2\delta+2\delta^2R^2+1}{4\delta}\end{cases} \leq 0,\end{aligned}$$

where the last inequality holds because $f_I(\cdot)\geq 0$ and

$$\frac{\partial\tilde{c}_3}{\partial\alpha} = \begin{cases}\left(-4\alpha\delta+2\delta+2\delta^2R^2+1\right)^{-\frac{1}{2}} & \text{if } \delta+1\geq\sqrt{-4\alpha\delta+2\delta+2\delta^2R^2+1},\\ 0 & \text{otherwise,}\end{cases} \geq 0.$$

Learning effect: Since the entrant wins the first stage auction for all $\alpha\geq 0$ if $c_I<\frac{1}{2}$, the publisher's Stage 2 payoff depends on α only for $c_I\geq\frac{1}{2}$. Therefore, $\frac{\partial(\pi_{S2}(\alpha)-\pi_{S2}(0))}{\partial\alpha}=\frac{\partial\pi_{S2}(\alpha)}{\partial\alpha}$, which in turn is equal to

$$\frac{\partial}{\partial\alpha}\left(\int_{c_I\geq\frac{1}{2},c_I b^*_{I1}\geq\mu_E b^*_{E1}(\alpha)}\frac{1}{2}dF_I+\int_{c_I\geq\frac{1}{2},c_I b^*_{I1}<\mu_E b^*_{E1}(\alpha)}\left(\int_0^{c_I}\max[c_E,R]\,dF_E+(1-F_E(c_I))c_I\right)dF_I\right),$$

which simplifies to

$$\frac{\partial}{\partial\alpha}\left(\int_{c_I\geq\frac{1}{2},c_I b^*_{I1}\geq\mu_E b^*_{E1}(\alpha)}\frac{1}{2}dF_I+\int_{c_I\geq\frac{1}{2},c_I b^*_{I1}<\mu_E b^*_{E1}(\alpha)}\left(\frac{R^2-c_I^2}{2}+c_I\right)dF_I\right). \tag{OA5}$$

Recall from the proof of Proposition 3 that there exists a unique $\bar{c}>\mu_E=\frac{1}{2}$ such that the entrant wins the first stage auction for all $c_I<\bar{c}$ and the incumbent wins for all $c_I\geq\bar{c}$. Therefore, we can find a threshold ad credit level $\tilde{\alpha}$ such that for all $\alpha>\tilde{\alpha}$, the entrant wins in Stage 1 for all c_I. For this range of α, the learning value would be zero because the entrant's CTR would have been revealed regardless of the incumbent's CTR. By monotonicity, the threshold $\tilde{\alpha}$ must occur at the α such that the effective bids of the two advertisers coincide for $c_I = 1$. Simple algebra yields $\tilde{\alpha} = \frac{1+\delta R^2}{2}$. Therefore,

$$(OA5) = \begin{cases} \frac{\partial}{\partial \alpha}\left(\int_{\tilde{c}_4}^{1} \frac{1}{2} n c_I^{n-1} dc_I + \int_{\frac{1}{2}}^{\tilde{c}_4} \left(\frac{R^2 - c_I^2}{2} + c_I\right) n c_I^{n-1} dc_I\right) & \text{if } \alpha \leq \frac{1+\delta R^2}{2}, \\ 0 & \text{if } \alpha > \frac{1+\delta R^2}{2}, \end{cases}$$

$$= \begin{cases} \frac{\partial}{\partial \alpha}\left(-\frac{1}{2}\tilde{c}_4^n + \left(\frac{R^2}{2}\tilde{c}_4^n - \frac{1}{2}\frac{n}{n+2}\tilde{c}_4^{n+2} + \frac{n}{n+1}\tilde{c}_4^{n+1}\right)\right) & \text{if } \alpha \leq \frac{1+\delta R^2}{2}, \\ 0 & \text{if } \alpha > \frac{1+\delta R^2}{2}. \end{cases}$$

where $\tilde{c}_4 \equiv \frac{2\delta - \sqrt{-4\alpha\delta + 2\delta + 2\delta^2 R^2 + 1} + 1}{2\delta}$. By explicitly solving the derivative in the top branch, it can be shown that the derivative is proportional to

$$2\alpha\delta - \delta + \sqrt{-4\alpha\delta + 2\delta + 2\delta^2 R^2 + 1} + \delta^2 R^2 - 1. \tag{OA6}$$

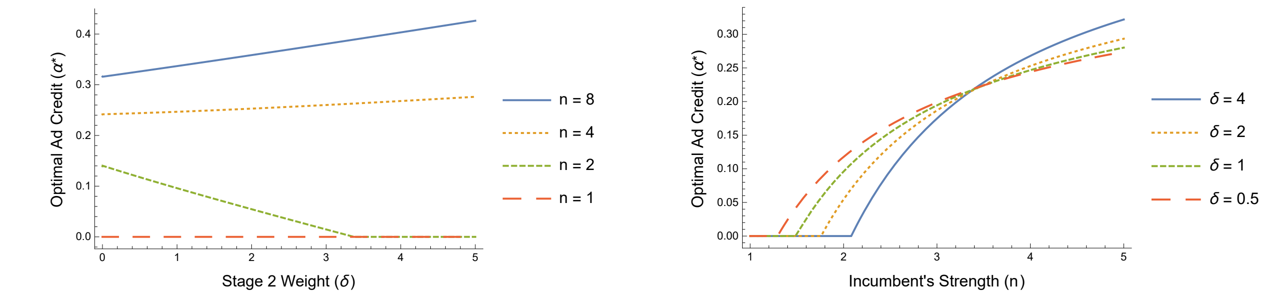
Therefore, using the fact that the (OA6) is increasing in R, we obtain the following learning effect pattern for general range of R: $(OA6)$ is positive if $\frac{1}{2} < \alpha \leq \frac{1+R^2\delta}{2}$ or $\alpha \leq \frac{1}{2}, R > r'$ and negative otherwise, where $r' = \frac{\sqrt{1+(1-2\alpha)\delta} - 1}{\delta}$. With the added assumption that $R \geq \frac{\sqrt{1+\delta}-1}{\delta}$, we obtain that the derivative is always positive for $\alpha \leq \frac{1+R^2\delta}{2}$ and zero for $\alpha > \frac{1+R^2\delta}{2}$. To establish convexity, we need only differentiate (OA6) with respect to α once more. This yields that the second derivative of the learning effect is $\frac{\partial(OA6)}{\partial \alpha} = 2\delta + \frac{1}{2}\left(-4\alpha\delta + 2\delta + 2\delta^2 R^2 + 1\right)^{-\frac{1}{2}}(-4\delta) \propto 1 - \left(-4\alpha\delta + 2\delta + 2\delta^2 R^2 + 1\right)^{-\frac{1}{2}} \geq 0$, where the last inequality follows from the fact that $\alpha \leq \frac{1+\delta R^2}{2} \implies -4\alpha\delta + 2\delta + 2\delta^2 R^2 + 1 \geq 1 \iff 1 - \left(-4\alpha\delta + 2\delta + 2\delta^2 R^2 + 1\right)^{-\frac{1}{2}} \geq 0$. ■

Optimal Ad Credit

A closed-form analytical expression for optimal α is not tractable with general CTR functional forms F_I and F_E. In this section, we provide numerical plots for specific functional forms and parameter values to better understand the forces that determine the publisher's optimal ad credit level.

Figure OA9 plots the optimal ad credit with respect to the Stage 2 weight parameter δ and the incumbent CTR strength parameter n. Figure OA9a reveals an interesting relationship between the optimal ad credit level and δ. When n is small such that the incumbent is likely to be weak, the publisher lowers the level of ad credit as δ increases ($n = 1$ and $n = 2$ in Figure OA9a). On the other hand, when n is large, then the ad credit increases with δ ($n = 4$ and $n = 8$ in Figure OA9a). This pattern can be understood based on the insights obtained in the main model.

Intuitively, when the incumbent is likely to be weak, the entrant is more likely to win in Stage 1. This



(a) Function of δ *(b) Function of n*

Figure OA9: Optimal Ad Credit; $R = \frac{1}{4}, F_E(c) = c, F_I(c) = c^n$

means that if the publisher offers ad credit, it is likely to be transferred to the entrant, costing the publisher money. Thus, the publisher lowers the ad credit level for smaller values of n. Furthermore, as δ increases, the entrant has stronger incentives to overbid to capitalize on the Stage 2 benefits of revealing its CTR. This increases the probability of the entrant winning in Stage 1. Therefore, the publisher lowers α^* accordingly as δ increases.

Next, consider the case when the incumbent is likely to be strong (i.e., n is large). Then, the entrant bids close to valuation, whereas the incumbent likely overbids (see Lemma 2). And since the magnitude of the incumbent's overbidding pattern increases with δ, the publisher anticipates that the entrant's CTR will likely be masked for high δ. In response, the publisher offers larger ad credit to the entrant (when n is large) as δ increases, in order to both learn the entrant's CTR more quickly and also extract more surplus from the incumbent.

In sum, we find two patterns of the optimal ad credit level which are consistent with the discussion of Proposition OA1. First, the higher the likelihood of the incumbent being strong, the larger the ad credit offered to the entrant. Second, this relationship is strengthened by the Stage 2 weight parameter δ; that is, if δ is high, then the search offers a much smaller (larger) ad credit when the incumbent is likely to be weak (strong). This latter phenomenon is depicted in Figure OA9b through the increasing steepness of the ad credit curve as δ increases.

OA2 Lemmas for GSP Extension

OA2.1 LREF Stage 2 Payoffs

Lemma OA1 (Assortative Matching)**.** *In any LREF equilibrium, the resulting ad slot allocation is assortative; i.e., the advertiser in position i has higher CTR than the advertiser in position $i+1$.*

Proof. Let $c_{(i)}$, $b_{(i)}$, and $p_{(i)}$, respectively, denote the CTR, bid, and payment of the advertiser in position i. The envy-free conditions are $\alpha_i c_{(i)} - p_{(i)} \geq \alpha_{i+1} c_{(i)} - p_{(i+1)}$ and $\alpha_{i+1} c_{(i+1)} - p_{(i+1)} \geq \alpha_i c_{(i+1)} - p_{(i)}$. Rearranging and combining the inequalities yield $(\alpha_i - \alpha_{i+1}) c_{(i)} \geq p_{(i)} - p_{(i+1)} \geq (\alpha_i - \alpha_{i+1}) c_{(i+1)}$, which together with the fact that $\alpha_i > \alpha_{i+1}$ imply $c_{(i)} \geq c_{(i+1)}$. ■

Lemma OA2 (LREF Stage 2 Payoffs)**.** *Let $I(i)$ denote the incumbent with the higher (lower) CTR. If the entrant's CTR is learned, then the advertisers' expected Stage 2 payoffs are*

$$\begin{aligned}
\mathbb{E}[\pi_i^l] &= F_E(c_i)(c_I - c_i) + \int_0^{c_i} \theta(c_i - c_E)\, dF_E + \int_{c_i}^{c_I} (c_I - c_E) + \theta(c_E - c_i)\, dF_E + (1 - F_E(c_I))\theta(c_I - c_i), \\
\mathbb{E}[\pi_i^l] &= \int_0^{c_i} \theta(c_i - c_E)\, dF_E, \\
\mathbb{E}[\pi_E^l] &= \int_{c_i}^{c_I} \theta(c_E - c_i)\, dF_E + \int_{c_I}^{1} (c_E - c_I) + \theta(c_I - c_i)\, dF_E.
\end{aligned}$$

If the entrant's CTR is not learned, then the expected payoffs are

	$\mu_E \leq c_i < c_I$	$c_i < \mu_E \leq c_I$	$c_i < c_I < \mu_E$
$\mathbb{E}[\pi_i^0]$	$(c_I - c_i) + \theta(c_i - \mu_E)$	$(c_I - \mu_E) + \theta(\mu_E - c_i)$	$\theta(c_I - c_i)$
$\mathbb{E}[\pi_i^0]$	$\theta(c_i - \mu_E)$	0	0
$\mathbb{E}[\pi_E^0]$	0	$\theta(\mu_E - c_i)$	$(\mu_E - c_I) + \theta(c_I - c_i)$

Proof. First, consider the advertisers in positions 2 and 3, where the "third position" is the no-display slot with a position-CTR of 0. The envy-free conditions imply $\alpha_2 c_{(2)} \left(1 - \frac{c_{(3)} b_{(3)}}{c_{(2)}}\right) \geq 0$ and $0 \geq \alpha_2 c_{(3)} \left(1 - \frac{c_{(3)} b_{(3)}}{c_{(3)}}\right)$, which simplify to $1 \leq b_{(3)} \leq c_{(2)}/c_{(3)}$. Note that there exist values of $b_{(3)}$ that satisfy this inequality because Lemma OA1 implies that $c_{(2)}/c_{(3)} \geq 1$. And since we choose the lowest revenue envy-free (LREF) equilibrium, we have that $b_{(3)} = 1$; i.e., the losing advertiser bids truthfully. Next, we find the LREF bid of advertiser (2) by recursion. Again, the envy-free conditions between advertisers (1) and (2) imply $\alpha_1 c_{(1)} \left(1 - \frac{c_{(2)} b_{(2)}}{c_{(1)}}\right) \geq \alpha_2 c_{(1)} \left(1 - \frac{c_{(3)} b_{(3)}}{c_{(1)}}\right)$ and $\alpha_2 c_{(2)} \left(1 - \frac{c_{(3)} b_{(3)}}{c_{(2)}}\right) \geq \alpha_1 c_{(2)} \left(1 - \frac{c_{(2)} b_{(2)}}{c_{(2)}}\right)$. Substituting $b_{(3)} = 1$ and simplifying yields

$$c_{(2)} - \frac{\alpha_2}{\alpha_1}(c_{(2)} - c_{(3)}) \leq c_{(2)} b_{(2)} \leq c_{(1)} - \frac{\alpha_2}{\alpha_1}(c_{(1)} - c_{(3)}). \tag{OA1}$$

Using Lemma OA1 and $\alpha_1 > \alpha_2$, it can be easily shown that the lower bound of (OA1) is indeed smaller than the upper bound; i.e., $c_{(2)} - \frac{\alpha_2}{\alpha_1}(c_{(2)} - c_{(3)}) < c_{(1)} - \frac{\alpha_2}{\alpha_1}(c_{(1)} - c_{(3)})$; that is, there exist values of $b_{(2)}$ that satisfy the envy-free conditions. In LREF equilibrium, we have $b_{(2)} = 1 - \alpha_2/\alpha_1 \left(1 - c_{(3)}/c_{(2)}\right)$.

The profits for advertisers (1), (2), and (3), respectively, are $\pi_{(1)} = \alpha_1 c_{(1)} \left(1 - \frac{c_{(2)} - \frac{\alpha_2}{\alpha_1}(c_{(2)} - c_{(3)})}{c_{(1)}}\right)$,

$\pi_{(2)} = \alpha_2 c_{(2)} \left(1 - \frac{c_{(3)}}{c_{(2)}}\right)$, and $\pi_{(3)} = 0$. Using these expressions, combined with the assumptions that $\alpha_1 = 1, \alpha_2 = \theta$, we can write the advertisers' payoffs when c_E is learned (i.e., entrant won in Stage 1):

	$c_E \leq c_i < c_I$	$c_i < c_E \leq c_I$	$c_i < c_I < c_E$
π_I	$(c_I - c_i) + \theta(c_i - c_E)$	$(c_I - c_E) + \theta(c_E - c_i)$	$\theta(c_I - c_i)$
π_i	$\theta(c_i - c_E)$	0	0
π_E	0	$\theta(c_E - c_i)$	$(c_E - c_I) + \theta(c_I - c_i)$

The payoffs when the c_E is not learned and is thus assigned μ_E follow immediately. ■

OA2.2 LREF Stage 1 Bids

We denote by $[a_1, a_2, a_3]$ the locally envy-free equilibrium candidate where advertiser a_1 gets the first slot, advertiser a_2 the second, and advertiser a_3 the third (null) slot.

Case 1: Suppose $\mu_E \leq c_i < c_I$.

(i) $[I, i, E]$

Envy-free (EF) condition for E: $\theta\mu_E(1 - \mu_E b_E/\mu_E) + \delta\mathbb{E}[\pi_E^l] \leq 0 \iff b_E \geq 1 + \frac{\delta}{\theta\mu_E}\mathbb{E}[\pi_E^l]$; EF condition for i: $\theta c_i(1 - \mu_E b_E/c_i) + \delta\mathbb{E}[\pi_i^0] \geq \delta\mathbb{E}[\pi_i^l] \iff b_E \leq \frac{c_i}{\mu_E} + \frac{\delta}{\theta\mu_E}(\mathbb{E}[\pi_i^0] - \mathbb{E}[\pi_i^l])$ and $\theta c_i(1 - \mu_E b_E/c_i) + \delta\mathbb{E}[\pi_i^0] \geq c_i(1 - c_i b_i/c_i) + \delta\mathbb{E}[\pi_i^0] \iff c_i b_i \geq (1-\theta)c_i + \theta\mu_E b_E$; EF condition for I: $c_I(1 - c_i b_i/c_I) + \delta\mathbb{E}[\pi_i^0] \geq \theta c_I(1 - \mu_E b_E/c_I) + \delta\mathbb{E}[\pi_i^0] \iff c_i b_i \leq (1-\theta)c_I + \theta\mu_E b_E$. Given b_E, there always exist non-negative b_i such that EF conditions are satisfied, because we need $(1-\theta)c_i + \theta\mu_E b_E \leq c_i b_i \leq (1-\theta)c_I + \theta\mu_E b_E$ but $c_i < c_I$. On the other hand, there does not always exist non-negative b_E that satisfies EF conditions: note that we must have

$$1 + \frac{\delta}{\theta\mu_E}\mathbb{E}[\pi_E^l] \leq \frac{c_i}{\mu_E} + \frac{\delta}{\theta\mu_E}(\mathbb{E}[\pi_i^0] - \mathbb{E}[\pi_i^l]) \tag{OA2}$$

for E to set envy-free bids. Using the fact that $\mathbb{E}[\pi_i^0] - \mathbb{E}[\pi_i^l] = \theta(c_i - \mu_E) - \int_0^{c_i} \theta(c_i - c_E)\, dF_E = \int_{c_i}^1 \theta(c_i - c_E)\, dF_E < 0$, it can be shown that the inequality (OA2) holds if and only if

$$\delta \leq \frac{\theta(c_i - \mu_E)}{\mathbb{E}[\pi_E^l] - (\mathbb{E}[\pi_i^0] - \mathbb{E}[\pi_i^l])}. \tag{OA3}$$

In summary, $[I, i, E]$ is an envy-free equilibrium if and only if δ is sufficiently small (OA3); the corresponding lowest revenue equilibrium bids are $b_E^* = 1 + \frac{\delta}{\theta\mu_E}\mathbb{E}[\pi_E^l]$ and $b_i^* = 1 - \theta + \frac{\theta\mu_E}{c_i}b_E^*$.

(ii) $[I, E, i]$

EF condition for i: $\theta c_i(1 - c_i b_i/c_i) + \delta\mathbb{E}[\pi_i^0] \leq \delta\mathbb{E}[\pi_i^l] \iff 1 + \frac{\delta}{\theta c_i}(\mathbb{E}[\pi_i^0] - \mathbb{E}[\pi_i^l]) \leq b_i$; EF condition for E: $\theta\mu_E(1-c_ib_i/\mu_E)+\delta\mathbb{E}[\pi_E^l]\geq 0 \iff b_i \leq \frac{\mu_E}{c_i}+\frac{\delta}{\theta c_i}\mathbb{E}[\pi_E^l]$ and $\theta\mu_E(1-c_ib_i/\mu_E)+\delta\mathbb{E}[\pi_E^l] \geq \mu_E(1-\mu_E b_E/\mu_E)+\delta\mathbb{E}[\pi_E^l] \iff \mu_E b_E \geq (1-\theta)\mu_E+\theta c_i b_i$; EF condition for I: $c_I(1-\mu_E b_E/c_I)+\delta\mathbb{E}[\pi_i^l] \geq \theta c_I(1-c_ib_i/c_I)+\delta\mathbb{E}[\pi_i^l] \iff \mu_E b_E \leq (1-\theta)c_I+\theta c_i b_i$. Given b_i, there always exist non-negative b_E such that EF conditions are satisfied, because we need $(1-\theta)\mu_E+\theta c_i b_i \leq \mu_E b_E \leq (1-\theta)c_I+\theta c_i b_i$ but $\mu_E < c_I$. On the other hand, there does not always exist non-negative b_i that satisfies EF conditions. To see this, note that we must have

$$1+\frac{\delta}{\theta c_i}(\mathbb{E}[\pi_i^0]-\mathbb{E}[\pi_i^l]) \leq \frac{\mu_E}{c_i}+\frac{\delta}{\theta c_i}\mathbb{E}[\pi_E^l] \tag{OA4}$$

for i to set envy-free bids. Again, using the fact that $\mathbb{E}[\pi_i^0]-\mathbb{E}[\pi_i^l]<0$, it can be shown that the inequality (OA4) holds if and only if

$$\delta \geq \frac{\theta(c_i-\mu_E)}{\mathbb{E}[\pi_E^l]-(\mathbb{E}[\pi_i^0]-\mathbb{E}[\pi_i^l])}. \tag{OA5}$$

In summary, $[I,E,i]$ is an envy-free equilibrium if and only if δ is sufficiently large (OA5); the corresponding lowest revenue equilibrium bids are $b_i^* = 1+\frac{\delta}{\theta c_i}(\mathbb{E}[\pi_i^0]-\mathbb{E}[\pi_i^l])$ and $b_E^* = 1-\theta+\frac{\theta c_i}{\mu_E}b_i^*$.

(iii) $[E,I,i]$

This cannot be an envy-free equilibrium because the EF conditions for I and E are

$$\theta c_I(1-c_ib_i/c_I)+\delta\mathbb{E}[\pi_i^l] \geq c_I(1-c_Ib_I/c_I)+\delta\mathbb{E}[\pi_i^l] \iff c_Ib_I \geq (1-\theta)c_I+\theta c_i b_i,$$
$$\mu_E(1-c_Ib_I/\mu_E)+\delta\mathbb{E}[\pi_E^l] \geq \theta\mu_E(1-c_ib_i/\mu_E)+\delta\mathbb{E}[\pi_E^l] \iff c_Ib_I \leq (1-\theta)\mu_E+\theta c_i b_i.$$

And since $\mu_E < c_I$, there does not exist any b_I for any b_i such that I and E are envy-free.

Case 2: Suppose $c_i < \mu_E \leq c_I$.

(i) $[I,i,E]$

This cannot be an envy-free equilibrium because the EF conditions for E and i are $\theta\mu_E(1-\mu_E b_E/\mu_E)+\delta\mathbb{E}[\pi_E^l] \leq \delta\mathbb{E}[\pi_E^0] \iff \theta\mu_E b_E \geq \theta\mu_E+\delta(\mathbb{E}[\pi_E^l]-\mathbb{E}[\pi_E^0])$ and $\theta c_i(1-\mu_E b_E/c_i)+\delta\mathbb{E}[\pi_i^0] \geq \delta\mathbb{E}[\pi_i^l] \iff \theta\mu_E b_E \leq \theta c_i-\delta\mathbb{E}[\pi_i^l]$, where we have used the fact that $\mathbb{E}[\pi_i^0]=0$ if $c_i<\mu_E$. Since $\mathbb{E}[\pi_E^l]-\mathbb{E}[\pi_E^0]\geq 0$ from (A.28), it follows that $\theta\mu_E+\delta(\mathbb{E}[\pi_E^l]-\mathbb{E}[\pi_E^0]) > \theta c_i-\delta\mathbb{E}[\pi_i^l]$. This implies that there does not exist any b_E such that E and i are envy-free.

(ii) $[I,E,i]$

EF conditions for i and E: $\theta c_i(1-c_ib_i/c_i)+\delta\mathbb{E}[\pi_i^0] \leq \delta\mathbb{E}[\pi_i^l] \iff \theta c_i-\delta\mathbb{E}[\pi_i^l] \leq \theta c_i b_i$ and $\theta\mu_E(1-c_ib_i/\mu_E)+\delta\mathbb{E}[\pi_E^l] \geq \delta\mathbb{E}[\pi_E^0] \iff \theta c_i b_i \leq \theta\mu_E+\delta(\mathbb{E}[\pi_E^l]-\mathbb{E}[\pi_E^0])$; EF conditions for

E and I: $c_I(1-\mu_E b_E/c_I)+\delta\mathbb{E}[\pi_i^l] \geq \theta c_I(1-c_i b_i/c_I)+\delta\mathbb{E}[\pi_i^l] \iff \mu_E b_E \leq (1-\theta)c_I+\theta c_i b_i$ and $\mu_E(1-\mu_E b_E/\mu_E)+\delta\mathbb{E}[\pi_E^l] \leq \theta\mu_E(1-c_i b_i/\mu_E)+\delta\mathbb{E}[\pi_E^l] \iff \mu_E b_E \geq (1-\theta)\mu_E+\theta c_i b_i$. Therefore, the lowest revenue envy-free equilibrium bids are $b_i^* = \max\left[1-\delta\int_0^{c_i} 1-\frac{c_E}{c_i}dF_E, 0\right]$ and $b_E^* = 1-\theta+\frac{\theta c_i}{\mu_E}b_i^*$.

(iii) $[E, I, i]$

This cannot be an envy-free equilibrium because the EF conditions for I and E are $c_I(1-c_I b_I/c_I)+\delta\mathbb{E}[\pi_i^l] \leq \theta c_I(1-c_i b_i/c_I)+\delta\mathbb{E}[\pi_i^l] \iff c_I b_I \geq (1-\theta)c_I+\theta c_i b_i$ and and $\mu_E(1-c_I b_I/\mu_E)+\delta\mathbb{E}[\pi_E^l] \geq \theta\mu_E(1-c_i b_i/\mu_E)+\delta\mathbb{E}[\pi_E^l] \iff c_I b_I \leq (1-\theta)\mu_E+\theta c_i b_i$. And since $\mu_E \leq c_I$, there does not exist any b_I for any b_i such that I and E are envy-free.

OA3 Entry May Reduce Incumbent's CTR

Suppose entry exogenously reduces the incumbent's CTR by a stochastic factor γ which has c.d.f. F_γ with support $[0,1]$.[36] We will show that the entrant overbidding and the weak (strong) incumbent under(over)bidding results carry over. We have

$$b_E^* = 1+\frac{\delta}{\mu_E}\int_0^1\left(\int_{\max[\gamma c_I,R]}^1 c_E-\max[\gamma c_I,R]\,dF_E-(\mu_E-\max[\gamma c_I,R])^+\right)dF_\gamma.$$

Since $(c_E-\max[\gamma c_I,R])^+$ is convex in c_E for all γ, it follows that regardless of the realization of γ, $\int_{\max[\gamma c_I,R]}^1 c_E-\max[\gamma c_I,R]\,dF_E-(\mu_E-\max[\gamma c_I,R])^+$ is always non-negative. This proves that the entrant always overbids. Similarly, whether the incumbent bids below or above valuation depends on the sign of

$$\int_0^1\left((\gamma c_I-\max[\mu_E,R])^+-\int_0^1(\gamma c_I-\max[c_E,R])^+\,dF_E\right)dF_\gamma.$$

Applying the machinery from the proof of Proposition 8 in Section 5.2.2, there exists a pair of thresholds such that the incumbent underbids for c_I below the low threshold and overbids for c_I above the high threshold.

[36] Since F_γ is general, this subsumes the case where entry reduces c_I by some deterministic factor.